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Abstract

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ON PERFECT MAPPINGS OF PARACOMPACT SPACES

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In the present paper we investigate properties of paracompact spaces admitting perfect mappings* onto metric spaces, and consider the question of approximating spaces by inverse spectra of metric spaces with perfect projections. The results obtained here adjoin the results of V. Ponomarev ([1,7]), B. Pasynkov ([2]), and A. Arkhangel' skii ([8]).

B. Pasynkov, generalizing the well-known theorems of S. Mardešić ([3]), proved in ([2]) a theorem on the factorization of continuous mappings into metric spaces. For perfect mappings the following assertion holds.

Theorem 1. *If for a paracompact space X there exists a perfect mapping $f : X \to R$ onto some metric space R, then for every cover ω* ***of the space*** *X* ***there exist a metric space*** *S* ***and perfect mappings*** *$g : X \to S$* ***and*** *$h : S \to R$* ***such that*** *g* ***is an*** *ω****-mapping*** and $f = hg$. If, moreover, $\dim X = n$, then for every cover ω of the space X there exist a metric space $\widetilde{S}$, whose dimension $\dim \widetilde{S} = n$, and perfect mappings $\tilde{g} : X \to \widetilde{S}$ and $\tilde{h} : \widetilde{S} \to R$, subject to the same conditions: $\tilde{g}$ is an ω-mapping and $f = \tilde{h}\tilde{g}$.*

For the proof we shall need the following

Lemma. *Let a perfect mapping $f : X \to R$ of a completely regular space X onto a completely regular space R be given, a mapping $g : X \to S$ onto a completely regular space S, and a mapping $h : S \to R$. Then, if $f = hg$, the mappings g and h are also perfect.*

Proof of the lemma. It is easy to see that the mapping h is perfect under our assumptions. The mapping g is bicompact, since the elements of the decomposition $\{g^{-1}z\}_{z\in S}$ are closed subsets of the elements of the decomposition $\{f^{-1}y\}_{y\in R}$. To prove the lemma it remains only to prove the closedness of the mapping g. Consider the mappings

$$\overline{f} : \beta X \to \beta R, \quad \overline{g} : \beta X \to \beta S \quad \text{and} \quad \overline{h} : \beta S \to \beta R,$$

which are extensions of the mappings f, g, and h to the maximal bicompact extensions βX and βS, so that $\overline{f}(\beta X \setminus X) = \beta R \setminus R$, $\overline{h}(\beta S \setminus S) = \beta R \setminus R$**** and $\overline{f} = \overline{h}\,\overline{g}$. To prove the closedness of the mapping g, it is enough to show that $\overline{g}(\beta X \setminus X) = \beta S \setminus S$. Suppose that this last equality does not hold. Then, since $\overline{h}(\beta S \setminus S) = \beta R \setminus R$ and $\overline{f} = \overline{h}\,\overline{g}$, the equality

$$\overline{f}(\beta X \setminus X) = \beta R \setminus R$$

would not hold. The lemma is proved.

Proof of Theorem 1. Take in the space R a sequence $\{\alpha_i\}$ of covers by balls of radius $1/2^i$, $i = 1, 2, ...$. The sequence of covers $\{\omega_i\}$ of the space X, where $\omega_i = f^{-1}\alpha_i$,

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* A continuous mapping $f : X \to R$ is called perfect if it is closed and bicompact in the sense that the inverse images $f^{-1}y$ of all points $y \in R$ are bicompact.

** Covers here are everywhere assumed to be open.

*** A continuous mapping $f : X \to S$ of a space X into some space S is called an ω-mapping for a given cover ω of the space X if for every point $y \in S$ there exists a neighborhood Oy whose inverse image $f^{-1}Oy$ is contained in at least one element of the cover ω.

**** These relations are fulfilled automatically, since f and h are perfect.

$i = 1, 2, ...,$ is a normal sequence.* Introduce the following notation. For any two covers α and β, denote by $\alpha \wedge \beta$ the cover consisting of all sets that are intersections of some element of the cover α with some element of the cover β.

We shall now construct by induction a normal sequence of covers $\{\gamma_i\}$, subject to the following conditions: 1) the cover γ_1 is inscribed in the cover ω, and 2) the cover γ_i is inscribed in the cover ω_i for each $i = 1, 2, ...$. We do this as follows. For $i = 1$, take the cover $\omega \wedge \omega_1$ as the cover γ_1. Suppose that for all $i = 1, 2, \dots, k-1$ covers γ_i have been constructed satisfying our conditions, and as the cover γ_k take any cover star-inscribed in $\gamma_{k-1} \wedge \omega_k$. Such a cover always exists, since X is paracompact.

The normal sequence $\{\gamma_i\}$ induces some metric space S and a continuous mapping $g : X \to S$ (see (1)). The sequence $\{\gamma_i\}$ is inscribed in the sequence $\{\omega_i\}$, and the elements of the partition $\{g^{-1}z\}_{z \in S}$ are subsets of the elements of the partition $\{f^{-1}y\}_{y \in R}$.

Define the perfect mapping $h : S \to R$ as follows:

$$h = fg^{-1}.$$

According to the lemma, the mapping $g : X \to R$ is perfect.

Let now $\dim X = n$. Into the normal sequence $\{\gamma_i\}$, as shown by B. A. Pasynkov, one can inscribe a normal sequence of covers $\{\mu_i\}$ subject to the following conditions:

1°. Each element of the cover μ_{i+1}, $i = 1, 2, ...$, intersects only a finite number of elements of the cover μ_i.

2°. The multiplicity of any cover μ_i does not exceed $n + 1$.

The space $\widetilde{S}$ induced by the sequence of covers $\{\mu_i\}$ has dimension $\dim \widetilde{S} = n$ (see (6)). The mapping $\tilde{g}$, induced by this sequence, will, as is easy to verify, be perfect. The mapping $\tilde{h} = \tilde{f}\tilde{g}^{-1}$ is also perfect. The theorem is proved.

Theorem 2. *A space X is a Čech-complete paracompactum if and only if for every cover ω of the space X there exists a perfect ω-mapping $f : X \to S$ onto some complete metric space S.*

Proof.

1°. **Necessity.** Z. Frolík proved (5) that for a Čech-complete paracompactum there exists a perfect mapping onto a complete metric space. Taking an arbitrary cover ω of the space X and applying Theorem 1, we obtain a metric space S and a perfect ω-mapping $f : X \to S$ onto the space S. The completeness of the space S follows from a result proved by V. Ponomarev (4), asserting the preservation of completeness in both directions under perfect mappings.

2°. Sufficiency follows directly from the fact that if for every cover ω of the space X there exists an ω-mapping onto some metric space S, then X is a paracompactum, and from the aforementioned result of V. Ponomarev. The theorem is proved.

Theorem 3. *If a paracompactum X admits a perfect mapping onto some metric space R, then X is the limit of some spectrum,*

$$X = \lim_{\leftarrow}\{X_\alpha, \mathfrak{D}_\alpha^\beta\},$$

whose elements X_α are metric spaces, and all projections $\mathfrak{D}_\alpha^\beta$ are perfect mappings.

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* A sequence of covers $\{\omega_i\}$, $i = 1, 2, ...$, is called **normal** if for each i the cover ω_{i+1} is star-inscribed in the cover ω_i, i.e., for each point $x \in X$ the union of the elements of the cover ω_{i+1} containing this point is contained in some element of the cover ω_i.

Proof. Let $\{\omega_\alpha\}_{\alpha\in\mathfrak{A}}$ be the set of all covers of the space X. To each cover ω_α we assign a certain metric space X_α and a mapping $\tilde{\omega}_\alpha : X \to X_\alpha$ which is a perfect ω-mapping. Denote by D the diagonal of the product $X \times X$. For each pair α_1, α_2 take the space

$$\overline{\omega}_{(\alpha_1\alpha_2)}D = \{\tilde{\omega}_{\alpha_1}(x), \tilde{\omega}_{\alpha_2}(x)\} = X_{(\alpha_1\alpha_2)} \subset X_{\alpha_1} \times X_{\alpha_2}.$$

Denote by h the natural mapping of the space X onto the space D, which assigns to each point $x \in X$ the point $(x, x) \in D$. The mapping $\tilde{\omega}_{(\alpha_1\alpha_2)} = \overline{\omega}_{(\alpha_1\alpha_2)}h$, according to the lemma, is a perfect mapping of the space X onto the space $X_{(\alpha_1\alpha_2)}$. The natural projections $\tilde{\omega}_{\alpha_1}^{(\alpha_1\alpha_2)}$ and $\tilde{\omega}_{\alpha_2}^{(\alpha_1\alpha_2)}$ of the space $X_{(\alpha_1\alpha_2)}$ onto the factor spaces X_{α_1} and X_{α_2} are also perfect mappings; $\tilde{\omega}_{\alpha_1} = \tilde{\omega}_{\alpha_1}^{(\alpha_1\alpha_2)}\tilde{\omega}_{(\alpha_1\alpha_2)}$ and $\tilde{\omega}_{\alpha_2} = \tilde{\omega}_{\alpha_2}^{(\alpha_1\alpha_2)}\tilde{\omega}_{(\alpha_1\alpha_2)}$.

Now denote by $D^{(2)}$ the diagonal of the product $D \times D$. For each pair $(\alpha_1\alpha_2)$, $(\alpha_3\alpha_4)$ put

$$\overline{\omega}_{(\alpha_1\alpha_2\alpha_3\alpha_4)}D^{(2)} = \{\tilde{\omega}_{(\alpha_1\alpha_2)}(x), \tilde{\omega}_{(\alpha_3\alpha_4)}(x)\} = X_{(\alpha_1\alpha_2\alpha_3\alpha_4)} \subset X_{(\alpha_1\alpha_2)} \times X_{(\alpha_3\alpha_4)}$$

and repeat the arguments given above.

Continuing this process, we obtain a spectrum $S = \{X_\alpha, \tilde{\omega}_\alpha^\beta\}$, whose elements X_α are metric spaces, and all projections $\tilde{\omega}_\alpha^\beta$ are perfect mappings (here the indices α denote sets $(\alpha_1\alpha_2 \dots \alpha_{2^n})$).

The mappings $\tilde{\omega}_\alpha : X \to X_\alpha$, defined for all indices $\alpha = (\alpha_1 \dots \alpha_{2^n})$ and satisfying the condition $\tilde{\omega}_\alpha = \tilde{\omega}_\alpha^\beta \tilde{\omega}_\beta$ for $\beta > \alpha$, determine a mapping $\tilde{\omega}$ of the space X into the space

$$\widetilde{X} = \varprojlim \{X_\alpha, \tilde{\omega}_\alpha^\beta\}$$

by assigning to each point $x \in X$ the point $\tilde{\omega}(x) = \{x_\alpha\}$, where $x_\alpha = \tilde{\omega}_\alpha(x)$. We shall show that the mapping $\tilde{\omega} : X \to \widetilde{X}$ is a homeomorphism. Let x' and x'' be two distinct points of the space X. For the cover $\omega = \{X \setminus x', X \setminus x''\}$ there exists $\alpha = \alpha(x', x'')$ for which $\tilde{\omega}_\alpha : X \to X_\alpha$ is an ω-mapping. It is clear that $\tilde{\omega}_\alpha(x') \neq \tilde{\omega}_\alpha(x'')$. Thus the mapping $\tilde{\omega} : X \to \widetilde{X}$ is one-to-one. We now show that for every open set $O \subset X$ the set $\tilde{\omega}(O)$ is open in $\widetilde{X}$. Take an arbitrary point $x \in O$. For the cover $\omega = \{O, X \setminus x\}$ there exists an α such that the mapping $\tilde{\omega}_\alpha : X \to X_\alpha$ will be an ω-mapping. Hence there exists a neighborhood $O_{\alpha y}$ of the point $y = \tilde{\omega}_\alpha(x)$ such that $\tilde{\omega}_\alpha^{-1}(O_{\alpha y}) \subseteq O$. Running through all points of the set O, we obtain that the set

$$\tilde{\omega}(O) = \tilde{\omega}\Big(\bigcup_\alpha \tilde{\omega}_\alpha^{-1}(O_{\alpha y})\Big) = \bigcup_\alpha \tilde{\omega}_\alpha^{-1}(O_{\alpha y}),$$

where by $\tilde{\omega}_\alpha$ are denoted the projections of the space $\widetilde{X}$ onto X_α. The mapping $\tilde{\omega}$ is open. Moreover, as is clear from the construction, the mapping $\tilde{\omega}$ is a mapping onto the whole space X. The theorem is proved.

Since both paracompactness and completeness in the sense of Čech are preserved in both directions under perfect mappings, Theorem 3 immediately implies

Theorem 4. *A space X is paracompact and complete in the sense of Čech if and only if it is the limit of some spectrum of complete metric spaces with perfect projections.*

We now consider the question of approximating n-dimensional, in the sense of dim, paracompacts by spectra of n-dimensional, in the sense of dim, metric spaces with perfect projections.

Theorem 5. *If a paracompact X of dimension $\dim X = n$ admits a perfect mapping $f : X \to R$ onto some metric space R, then the space X is the limit of some spectrum*

$$X = \varprojlim \{R_\alpha, \tilde{\omega}_\alpha^\beta\}$$

of n-dimensional in the sense of dim *metric spaces R_α, and all projections $\tilde{\omega}_\alpha^\beta$ are perfect mappings.*

From this theorem it follows at once

Theorem 6. *A Čech-complete paracompact space X of dimension $\dim X = n$ is the limit of a certain spectrum of n-dimensional, in the sense of* dim*, complete metric spaces with perfect projections.*

Proof of Theorem 5. Let $\{\omega_\alpha\}_{\alpha\in\Omega}$ be the set of all covers of the space X. To each cover ω_α we assign an n-dimensional, in the sense of dim, metric space R_α and a perfect ω_α-mapping $\mathfrak{F}_\alpha : X \to R_\alpha$. For every pair α_1, α_2 there exists a perfect mapping

$$\overline{\mathfrak{F}}_{(\alpha_1\alpha_2)} : X \to R_{\alpha_1} \times R_{\alpha_2},$$

defined as follows:

$$\overline{\mathfrak{F}}_{(\alpha_1\alpha_2)}(x) = (\mathfrak{F}_{\alpha_1}(x), \mathfrak{F}_{\alpha_2}(x))$$

for all $x \in X$. By Theorem 1 there exists an n-dimensional, in the sense of dim, metric space $R_{(\alpha_1\alpha_2)}$ and perfect mappings

$$\mathfrak{F}_{(\alpha_1\alpha_2)} : X \to R_{(\alpha_1\alpha_2)}$$

and

$$h_{(\alpha_1\alpha_2)} : R_{(\alpha_1\alpha_2)} \to R_{\alpha_1} \times R_{\alpha_2},$$

such that

$$\overline{\mathfrak{F}}_{(\alpha_1\alpha_2)} = h_{(\alpha_1\alpha_2)}\mathfrak{F}_{(\alpha_1\alpha_2)}.$$

Let us now denote by p_{α_i} the natural projections onto the factor spaces R_{α_i}, and put

$$\mathfrak{F}_{\alpha_1}^{(\alpha_1\alpha_2)} = p_{\alpha_1} h_{(\alpha_1\alpha_2)}, \qquad \mathfrak{F}_{\alpha_2}^{(\alpha_1\alpha_2)} = p_{\alpha_2} h_{(\alpha_1\alpha_2)}.$$

Obviously,

$$p_{\alpha_1}\overline{\mathfrak{F}}_{(\alpha_1\alpha_2)} = \mathfrak{F}_{\alpha_1} = \mathfrak{F}_{\alpha_1}^{(\alpha_1\alpha_2)}\mathfrak{F}_{(\alpha_1\alpha_2)}, \qquad p_{\alpha_2}\overline{\mathfrak{F}}_{(\alpha_1\alpha_2)} = \mathfrak{F}_{\alpha_2} = \mathfrak{F}_{\alpha_2}^{(\alpha_1\alpha_2)}\mathfrak{F}_{(\alpha_1\alpha_2)}.$$

Next take, for each pair $(\alpha_1\alpha_2)$, $(\alpha_3\alpha_4)$, the product $R_{(\alpha_1\alpha_2)} \times R_{(\alpha_3\alpha_4)}$ and repeat the preceding arguments. Continuing this process, we obtain a spectrum

$$S = \{R_\alpha, \mathfrak{F}_\alpha^\beta\},$$

where the indices α denote collections $(\alpha_1 \dots \alpha_{2n})$. Repeating the arguments of Theorem 3, we conclude that

$$X = \varprojlim \{R_\alpha, \mathfrak{F}_\alpha^\beta\}.$$

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.