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Abstract

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MATHEMATICS

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THE PROBLEM OF ADDING PRIME AND "ALMOST" PRIME NUMBERS IN ALGEBRAIC NUMBER FIELDS

(Presented by Academician I. M. Vinogradov on 15 VI 1964)

Let K be a field of algebraic numbers of degree $n = r_1 + 2r_2$ over the field of rational numbers; r_1 is the number of real conjugate fields and r_2 the number of pairs of complex conjugate fields for the field K. Let, further, d be the discriminant, R the regulator, h the number of ideal classes, and w the number of roots of unity of the field K.

Various authors have studied additive problems in algebraic fields, for example ([1–5]).

In the paper ([5]), A. I. Vinogradov extended the lower estimates obtained by A. Selberg' s sieve method to arbitrary algebraic fields and, in particular, proved that every sufficiently large in norm "even"integer ζ of the field K is representable in the form

$$\zeta = \alpha + \beta, \tag{A}$$

where the principal ideal (α) has no more than two prime divisors, and (β) has no more than three prime divisors. Moreover, the number of solutions of equation (A) has the lower estimate

$$C \frac{w^2 |d|^{3/2} \mathfrak{S}_k(\zeta)}{2^n (2\pi)^{r_2} (Rh)^2 \ln^2 |N\zeta|} |N\zeta|,$$

where $\mathfrak{S}_k(\zeta)$ is a special series, and C is an absolute constant.

In this note the following is proved.

Theorem 1. Every sufficiently large in norm "even" integer ζ of the field K is representable in the form

$$\zeta = \alpha + \beta, \tag{1}$$

where the principal ideal (α) is prime, and the principal ideal (β) is the product of no more than four prime ideals.

By the word "even" one should understand the following: if in K there exists an ideal $\mathfrak{p}$ with $N\mathfrak{p} = 2$, then $\mathfrak{p} \mid (\zeta)$. If, however, there is no such ideal in K, then ζ is any sufficiently large in norm integer of K.

In proving this theorem we use A. Selberg' s sieve according to the scheme proposed by B. V. Levin in ([6]), generalizing it to the field K.

We consider the sequence

$$a_n = \zeta - \xi_n, \tag{2}$$

where ξ_n runs through all prime numbers of the field K under the condition

$$N(a_n) \leq |N(\zeta)| = N. \tag{3}$$

By a prime number ξ we mean that the principal ideal (ξ) is prime; $N(\alpha)$ is the norm of the number α.

Let $z < N$, $y < z$, $r = \left[\frac{\ln N}{\ln y}\right] - 2$, and let Q be an integer rational number under the condition $0 \leq Q \leq r$; $A_Q(\zeta, z, y)$ is the number of terms of the sequence

(2), whose principal ideals are not divisible by any prime ideal $\mathfrak{p}$ with $N\mathfrak{p} < y$, and which have no more than Q prime factors $\mathfrak{q}$ with $y \leq N\mathfrak{q} < z$.

Let, further, $N_\mathfrak{n}$ denote the number of terms of the sequence (2) whose principal ideals are divisible by the ideal $\mathfrak{n}$; let $f(\mathfrak{n})$ be a certain multiplicative function,

$$N_\mathfrak{n} = \frac{N \cdot f(\mathfrak{n})}{N\mathfrak{n}} + R_\mathfrak{n}, \qquad R(x, y) = \sum_{\substack{\mathfrak{n} \mid \pi_x \\ N\mathfrak{n} \leq y}} \frac{f(\mathfrak{n})}{N\mathfrak{n} \prod_{\mathfrak{p} \mid \mathfrak{n}} \left(1 - \frac{f(\mathfrak{p})}{N\mathfrak{p}}\right)},$$

where

$$\pi_x = \prod_{N\mathfrak{p} < x} \mathfrak{p}.$$

Arguing in the same way as in [6], we obtain the fundamental inequality for estimating from below $A_Q(\xi, z, y)$, namely:

$$A_Q(\xi, z, y) \geq N\left[1 - \sum_{N\mathfrak{p}<y} \frac{f(\mathfrak{p})}{N\mathfrak{p}\, R(N\mathfrak{p}, y_\mathfrak{p})} - \frac{1}{Q+1} \sum_{y \leq N\mathfrak{p}<z} \frac{f(\mathfrak{p})}{N\mathfrak{p}\, R(y, y_\mathfrak{p})}\right] + \sum_{N\mathfrak{p}<y} \sum_{\substack{\mathfrak{n}|\pi_\mathfrak{p} \\ N\mathfrak{n}\leq y_\mathfrak{p}^2}} R_{\mathfrak{n}\mathfrak{p}} \gamma_{\mathfrak{n},\mathfrak{p}} + \frac{1}{Q+1} \sum_{\substack{y\leq N\mathfrak{p}<z \\ N\mathfrak{n}\leq y_\mathfrak{p}^2}} R_{\mathfrak{n}\mathfrak{p}} \gamma_{\mathfrak{n},\mathfrak{p}}. \tag{4}$$

Here

$$\gamma_{\mathfrak{n},\mathfrak{p}} = \sum_{\delta|\mathfrak{n}} \mu(\delta)\beta_{\delta,\mathfrak{p}}, \qquad \beta_{\delta,\mathfrak{p}} = \left(\sum \lambda_{\delta,\mathfrak{p}}\right)^2; \quad \lambda_{\delta,\mathfrak{p}} = 1, \quad \lambda_{\delta,\mathfrak{p}} = 0, \text{ if } \delta \neq \mathfrak{r};$$

$N\delta > y_\mathfrak{p}$; $y_\mathfrak{p} = \sqrt{y^\theta/N\mathfrak{p}}$; θ is an arbitrary real number; $\pi_\mathfrak{p}$ is the product of all prime ideals whose norms do not exceed $N\mathfrak{p}$.

The calculation of the main term in (4) is carried out in the same way as in [6], with specially chosen y and $f(\mathfrak{p}) = \dfrac{N\mathfrak{p}}{N\mathfrak{p}-1}$. Let us dwell in more detail on the estimate of the remainder term in (4)

$$R = \sum_{N\mathfrak{p}<y} \sum_{\substack{\mathfrak{n}|\pi_\mathfrak{p} \\ N\mathfrak{n}\leq y_\mathfrak{p}^2}} R_{\mathfrak{n}\mathfrak{p}} \gamma_{\mathfrak{n},\mathfrak{p}} + \frac{1}{Q+1} \sum_{\substack{y\leq N\mathfrak{p}<z \\ N\mathfrak{n}\leq y_\mathfrak{p}^2}} R_{\mathfrak{n}\mathfrak{p}} \gamma_{\mathfrak{n},\mathfrak{p}}. \tag{5}$$

Theorem 2. Let $\mathfrak{f}$ be an integral ideal in K; let H be an arbitrary class of ideals modulo $\mathfrak{f}$; let $h(\mathfrak{f})$ be the number of ideal classes modulo $\mathfrak{f}$; and let $\pi_K(x, H)$ be the number of prime ideals of the class H whose norms do not exceed x, $D = |d|N\mathfrak{f}$. Then

$$\sum_{D\leq x^{1/3}} \mu^2(\mathfrak{f}) \max_H \left|\pi_K(x, H) - \frac{\operatorname{li} x}{h(\mathfrak{f})}\right| \ll \frac{x}{\ln^A x}, \tag{6}$$

where A is any constant of the field K.

For the proof of Theorem 2, an essential role is played by the density theorem on zeros of Hecke L-series, which we formulate in the form of the main lemma.

Main Lemma. Let $0 < a < 1$, and let $N_K(\alpha, T)$ be the number of zeros of all $h(\mathfrak{f})$ Hecke L-functions of the field K with characters modulo $\mathfrak{f}$ in the rectangle $\alpha \leq \sigma \leq 1$, $|t| \leq T$, $s = \sigma + it$. Then the estimate

$$N_K(\alpha, T) \ll T^{1+4c_1(1-\alpha)} D^{3(1-\alpha)} \ln^{c_2} DT, \tag{7}$$

holds, where c_1 and c_2 are constants of the field K, $c_2 = 6n + 7$.

Proof. From Lemma 1 of paper [3], with $\eta = \frac{1}{\ln D}$, we obtain for the Hecke L-function $L_K(s, \chi)$

$$L_K\left(-\frac{1}{\ln D} + iT, \chi\right) \ll T^{3n/4} D^{1/4} \ln^n D, \tag{8}$$

$$L_K\left(1 + \frac{1}{\ln D} + iT, \chi\right) \ll \ln^n D. \tag{9}$$

By the three-circles theorem, from (8) and (9) we obtain

$$L_K\left(1/2 + iT, \chi\right) \ll T^{3n/4} D^{1/4} \ln^n D. \tag{10}$$

Applying estimate (10) and arguing in the same way as in paper [7], we obtain the validity of the main lemma.

The proof of Theorem 2 from the main lemma is obtained according to the scheme set out in paper [8], with minor modifications connected with passing from the field of rational numbers to the field K.

Now put $y = z^{\beta}$ in (4), with $0 < \beta \ll 1$. In the same way as in [6], we obtain

$$|R| \ll \sum_{|d| N\mathfrak{n} \le z^{\theta\beta}} \mu^2(\mathfrak{n}) \left| N\mathfrak{n} - \frac{N \cdot f(\mathfrak{n})}{N\mathfrak{n}} \right| \ln^3 z. \tag{11}$$

Choosing $z^{\theta\beta}$ so that $|R| \ll \frac{N}{\ln^A N}$, which is possible by Theorem 2 when

$$z^{\theta\beta} = N^{1/3},$$

we obtain the assertion of Theorem 1. Details in the case when K is the field of rational numbers are set out in paper [6].

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Note: Figure translations are in progress. See original paper for figures.