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INVESTIGATION OF THE ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE IN THE NONRELATIVISTIC THREE-BODY PROBLEM

(Presented by Academician N. N. Bogolyubov, 14 V 1964)

1. Investigation of the analytic properties of relativistic amplitudes indicates that scattering amplitudes in the complex plane of the orbital angular momentum may, in addition to poles, possess other singularities, in particular moving branch points, which can exert a decisive influence on the asymptotic behavior at high energies ([1]). The principal cause of such behavior is the influence of multiparticle intermediate states in relativistic theory. In view of the absence of rigorous proofs of these assumptions in the relativistic case, the study of nonrelativistic problems with multiparticle intermediate states, where exact equations for the scattering amplitude are available—in particular, the quantum-mechanical three-body problem—is of considerable interest.

An attempt to study the analytic properties of the scattering amplitude for the three-body problem in the plane of complex orbital angular momentum was undertaken in Newton' s work ([2a,b]), where he arrives at the conclusion that an infinite set of moving cuts exists in the plane of the complex orbital angular momentum. However, this result may not be entirely correct owing to the ambiguity of continuing the angular-momentum quantum number into the complex plane and to the ambiguity present in the Lippmann–Schwinger equation for the three-body problem.

To avoid these difficulties, Omnès ([3]) starts from the Faddeev equations ([4]), written for partial waves, introducing only the total orbital angular momentum of the whole system. The positive conclusion of this work is that the scattering amplitude on a bound state is meromorphic in the plane of the complex orbital angular momentum if the solution of the extended Faddeev equations is meromorphic ([3]), which in itself constitutes a rather difficult mathematical problem. Consequently, the question of the meromorphy of the scattering amplitude remains open.

The aim of the present work is to study the analytic properties of the scattering amplitude for the three-body problem when the scattering of a free particle by the bound state of the other two particles takes place.

2. Let us consider the problem of scattering of a particle of mass m_1 by the bound state of particles with masses m_2 and m_3. Let the mass m_3 be infinite, and let the interaction be effected by two-particle potentials, so that the Hamiltonian of the system has the form:

$$H = K_1 + K_2 + V_{13}(r_1) + V_{12}(|\mathbf{r}_1 - \mathbf{r}_2|) + V_{23}(r_2), \qquad H_0 = K_1 + K_2, \quad (2,1)$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the radius vectors of the corresponding particles relative to the force center, and K_1 and K_2 are their kinetic energies. The matrix element determining the probability of scattering by the bound state is written in the form:

$$R_{fi} = \left(\Phi_i,\ \widetilde{T}_{13}\Psi_i^{(3)} + \widetilde{T}_{12}\Psi_i^{(2)}\right), \quad (2,2)$$

where $\widetilde{T}_{13}$ and $\widetilde{T}_{12}$ are determined from the equations

$$\widetilde{T}_{ij}(z) = V_{ij} - V_{ij}G_{23}(z)\widetilde{T}_{ij}(z), \qquad i,j = 1,2;\ 1,3, \quad (2.3)$$

$G_{23}(z)$ is the Green' s function

$$G_{23}(z) = (H_0 + V_{23}(r_2) - z)^{-1}, \qquad z = E + i\eta \quad (\eta \to 0). \quad (2.4)$$

E is the total energy of the system, and the functions $\Psi_i^{(2,3)}$ satisfy the modified Faddeev equations

$$\begin{aligned} \Psi_i^{(2)} &= -\Phi_i - G_{23}(z)\widetilde{T}_{13}(z)\Psi_i^{(3)}, \\ \Psi_i^{(3)} &= -\Phi_i - G_{23}(z)\widetilde{T}_{12}(z)\Psi_i^{(2)}, \\ \Psi_i^{(1)} &= \Phi_i; \end{aligned} \quad (2.5)$$

Φ_i and Φ_f are the functions of the initial and final states of the system, respectively:

$$\Phi_i = (2\pi)^{-3/2}e^{i\mathbf{k}_i\mathbf{r}_1}\varphi_{n_i}(\mathbf{r}_2); \qquad \Phi_f = (2\pi)^{-3/2}e^{i\mathbf{k}_f\mathbf{r}_1}\varphi_{n_f}(\mathbf{r}_2). \quad (2.6)$$

Here $\mathbf{k}_f$ ($\mathbf{k}_i$) is the momentum of the incident particle after (before) scattering, and φ_{n_f} (φ_{n_i}) is the wave function of the bound state of the second particle after (before) scattering.

To study the analytic properties of the scattering amplitude in the plane of complex orbital angular momentum, we expand equations (2.5), written in the momentum representation, in partial waves. Let l_2 be the orbital angular momentum of the bound state of the second particle relative to the center, and l_1 the orbital angular momentum of the first particle relative to the same center. The total orbital angular momentum is $j = l_1 + l_2$. Then, instead of (2.5), we shall have

$$\chi^{(i)J}_{l_1 l_2; l_1' l' 2}(pn; k_0 n_0) = f^{(i)J}_{l_1 l_2; l_1' l' 2}(pn; k_0 n_0) + \frac{1}{D(q, E_{nl_2}, z)} \sum_{\nu_1 \nu_2} \left(\sum_{n''} + \int \right) \times$$

$$\times \int_0^\infty p''^2 dp''\, K^{(i)J}_{l_1 l_2; \nu_1 \nu_2}(pn\, p'' n''; n_0 k_0)\, \chi^{(i)J}_{\nu_1 \nu_2; l_1' l' 2}(p'' n''; k_0 n_0) \quad (i = 2, 3), \tag{2.7}$$

where K and f are defined functions (see ([6])), and

$$D(p, E_{nl_2}, z) = \frac{p^2}{2m_1} + E_{nl_2} - z. \tag{2.8}$$

Recall that the scattering amplitude is expressed through the functions $\chi^{(i)J}$ in a known manner ([6]).

In order to study the analytic properties of the solutions of equations (2.7) in the complex J-plane, it is first of all necessary to find the correct analytic continuation of the kernels of the integral equation. A formal solution of these equations may lead to dubious results, analogous to those given in ([2]). Indeed, the formal solution of equation (2.7) may be represented in the form

$$\chi^{(i)J}_{l_1 l_2; l_1' l' 2}(pn; k_0 n_0) = \frac{N^{(i)J}_{l_1 l_2; l_1' l' 2}(pn; k_0 n_0)}{D^{(i)J}(z)}, \tag{2.9}$$

where N and D are the first and second Fredholm determinants. In the lowest order of perturbation theory for D we obtain:

$$\sum_{l_2=0}^{\infty} \sum_{\tau=-l_2}^{l_2} \int_{a(-\infty)}^{a(z)} \frac{\rho^J_{\tau l_2}(\xi)}{J + \tau - \xi}\, d\xi, \tag{2.10}$$

where $a(z)$ is the position of the Regge pole for the two-particle amplitude t_{13}. Hence it is seen that, in the J-plane, the functions $\chi^{(i)J}$, and together with them the amplitudes scattering will have infinitely many moving cuts. However, these results may prove to be incorrect for the following reasons.

1°. The "natural" method of analytic continuation of angular momenta chosen by Newton is not unique. This difficulty can be avoided if, instead of expanding in the functions proposed by Newton, one uses an expansion in Wigner D^J-functions ([7], [3]).

2°. Under continuation into the complex domain in J, the averaged interaction contained in the kernel is an analytic function of J only under the condition

$$\operatorname{Re} J > -3/2 + l_2 + l_2'.$$

Since $0 \le l_2' \le \infty$, the domain of analyticity is practically reduced to zero.

3°. Restricting oneself only to the first term of the expansion in the Fredholm series may prove insufficient and, possibly, taking account of the subsequent terms of the expansion will substantially change the picture. Since at present a general investigation of the solution of equations (2.7) is very difficult, in the next paragraph we shall formulate a perturbation-theory method for the Faddeev equations and analyze individual terms of the series.

3. Using equations (2.2), one can obtain the following expansion in a perturbation-theory series:

$$R_{if} = \langle \Phi_f | T_{12} + T_{13} - T_{12} G_0 T_{13} - T_{13} G_0 T_{12} + T_{12} G_0 T_{13} G_0 T_{12} + \cdots | \Phi_i \rangle, \tag{3.1}$$

where

$$T_{ij} = V_{ij} - V_{ij} G_0(z) T_{ij}(z), \tag{3.2}$$

and G_0 is the free Green' s function

$$G_0(z) = (H_0 - z)^{-1}. \tag{3.3}$$

It is convenient to associate with each term of the expansion (3.1), taken in the momentum representation, a definite diagram (see also ([6])):

$$R_{if} = \;\_\;\_\; + \;\_*\_\; + \;\_\;*\_\; + \;\_*\;\_\; + \;\_\;*\;\_\; + \;\_\;\_\; + \;\_**\_\; + \;\_*\,*\_\; + \cdots + \tag{3.4}$$

[[unclear: diagrammatic line symbols in (3.4) reproduced only schematically]]

The upper horizontal line corresponds to the first particle, the lower one to the second; a circle corresponds to the scattering matrix t_{12}, and a cross on the upper or lower line denotes t_{13} or t_{23}, respectively, where $t_{ik}(z)$ are the two-particle scattering amplitudes off the energy shell. A vertical section of two intermediate lines corresponds to the free Green' s function $G_0(z)$. As examples we give the explicit expression for two diagrams. Taking into account that

$$(\mathbf{k}_f \mathbf{p}_f | T_{12}(z) | \mathbf{k}_i \mathbf{p}_i) =$$

$$= \delta(\mathbf{p}_f + \mathbf{k}_f - \mathbf{k}_i - \mathbf{p}_i) \left(\frac{\mathbf{p}_f - \mathbf{k}_f}{2} \middle| t_{12} \left(z - \frac{(\mathbf{p}_f + \mathbf{k}_f)^2}{4m_{12}} \right) \middle| \frac{\mathbf{p}_i - \mathbf{k}_i}{2} \right),$$

$$(\mathbf{k}_f \mathbf{p}_f | T_{13}(z) | \mathbf{k}_i \mathbf{p}_i) = \delta(\mathbf{p}_f - \mathbf{p}_i) \left(\mathbf{k}_f \middle| t_{13} \left(z - \frac{p_f^2}{2m_1} \right) \middle| \mathbf{k}_i \right), \tag{3.5}$$

$$(\mathbf{k}_f \mathbf{p}_f | T_{23}(z) | \mathbf{k}_i \mathbf{p}_i) = \delta(\mathbf{k}_f - \mathbf{k}_i) \left(\mathbf{p}_f \middle| t_{23} \left(z - \frac{k_f^2}{2m_2} \right) \middle| \mathbf{p}_i \right),$$

where m_{12} is the reduced mass, we have

$$\underline{-*-} = \int d\mathbf{p}\, \chi^*_{n_f}(\mathbf{p}) \chi_{n_i}(\mathbf{p}) \left(\mathbf{k}_f \middle| t_{13} \left(z - \frac{p^2}{2m_1} \right) \middle| \mathbf{k}_i \right), \tag{3.6}$$

$$-*- = \int \chi^*_{n_f}(\mathbf{p}_f) \chi_{n_i}(\mathbf{p}_i)\, d\mathbf{p}_f d\mathbf{p}_i \int d\mathbf{q} \left(\frac{\mathbf{k}_f - \mathbf{p}_f}{2} \middle| t_{12} \left(z - \frac{(\mathbf{p}_f + \mathbf{k}_f)^2}{4m_{12}} \right) \middle| \frac{\mathbf{p}_f + \mathbf{k}_f}{2} - \mathbf{q} \right);$$

$$\times \frac{1}{q^2 + (\mathbf{p}_f + \mathbf{k}_f - \mathbf{q})^2 - z} \left\langle \mathbf{p}_f + \mathbf{k}_f - \mathbf{q} \middle| t_{12} \left(z - \frac{q^2}{m_1} \right) \middle| \mathbf{p}_i + \mathbf{k}_i - \mathbf{q} \right\rangle \frac{1}{q^2 + (\mathbf{p}_i + \mathbf{k}_i - \mathbf{q})^2 - z} \times$$

$$\times \left\langle \frac{\mathbf{p}_i + \mathbf{k}_i}{2} - \mathbf{q} \middle| t_{12} \left(z - \frac{(\mathbf{p}_i + \mathbf{k}_i)^2}{4m_{12}} \right) \middle| \frac{\mathbf{k}_i + \mathbf{p}_i}{2} \right\rangle, \tag{3.7}$$

where $\chi_n(\mathbf{p})$ is the wave function of the bound state in the momentum representation. Consider the subclass of diagrams from (3.4) of the form:

$$-\times- \quad + \quad \begin{matrix} -\times-\times- \\ -\times- \end{matrix} \quad + \quad \begin{matrix} -\times-\times-\times- \\ -\times-\times- \end{matrix} \quad + \; \ldots \; + \tag{3.8}$$

For simplicity we shall assume that the bound state is an s-state with binding energy ε_0. For the first diagram of the series (3.8), at large $t = (\mathbf{k}_f - \mathbf{k}_i)^2$, we obtain an asymptotic expression of the type

$$\int f(p^2)\, t^\alpha (z - p^2)\, dp, \tag{3.9}$$

corresponding to a cut in the complex plane of the angular momentum. The study of the following graphs of the series (3.8) likewise leads to the conclusion that cuts exist in the complex angular-momentum plane.

However, the series (3.8) is easily summed. It is the expansion of the expression

$$(\Phi_f,\, T_{13}(1 - G_0 T_{23} G_0 T_{13})^{-1} \Phi_i). \tag{3.10}$$

On the other hand, it is easy to show that this expression is identically equal to

$$(\Phi_f, \widetilde{T}_{13}(z)\Phi_i).$$

Furthermore, it is clear that

$$(\Phi_f, \widetilde{T}_{13}\Phi_i) = \langle \mathbf{k}_f | t_{13}(z - \varepsilon_0) | \mathbf{k}_i \rangle. \tag{3.11}$$

But $\langle \mathbf{k}_f | t_{13}(z - \varepsilon_0) | \mathbf{k}_i \rangle$ is the exact scattering amplitude of the first particle by the force center, which, as is known, is a meromorphic function of the orbital angular momentum. Therefore the series (3.8) is also a meromorphic function of the orbital angular momentum. The result obtained is natural, since for $V_{12} = 0$ the Hamiltonian (2.1) admits separation of variables.

The example considered is interesting in the sense that each term of the perturbation-theory series may have a cut, whereas the series as a whole is a meromorphic function.

For a more detailed exposition of the questions touched upon here, see (6). In conclusion I express my deep gratitude to N. N. Bogolyubov and A. A. Logunov for discussions, and also to B. A. Arbuzov, A. V. Efremov, I. T. Todorov, and O. A. Khrustalev for fruitful discussions.

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Note: Figure translations are in progress. See original paper for figures.