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Abstract

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Mathematics

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On the ε-Entropy of Some Compact Sets of Differentiable Functions in Spaces with a Monotone Norm

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The study of the asymptotic behavior of quantities characterizing the massiveness of compact sets in functional spaces constitutes a natural circle of problems in the theory of functions, indicated by A. N. Kolmogorov (^{1}). In the present paper it is shown that, in these problems, results and methods from the domain of embedding theorems can be applied successfully. Here "almost sharp" estimates are obtained for the ε-entropy of a broad class of compact sets in functional Banach spaces with an arbitrary monotone translation-invariant norm. For the case of the metric C, these estimates, with some refinements, were obtained in the works of A. N. Kolmogorov and V. M. Tikhomirov (^{2}) and K. I. Babenko (^{3}). For the case of the metric L_p, the lower estimate (7) obtained below for integral s follows from the work of B. S. Mityagin (^{4}). It should also be noted that compact sets of the type $B(M, s)$ introduced below were studied earlier by N. M. Korobov (^{5}) in connection with cubature formulas; moreover, there is an interesting parallelism between results on ε-entropy and on cubature formulas.

Let $\|\cdot\|$ be some monotone translation-invariant norm of functions of n variables. It is easy to show that every such norm on the set of functions finite with respect to a fixed cube Ω satisfies, for some l, the inequalities

$$\|u\| \leq C\|D^l u\|_{L_2}, \qquad D^l = \partial^{ln}/\partial x_1^l \cdots \partial x_n^l; \tag{1}$$

$$\|u\|_{L_2} \leq C\|D^l u\| \tag{2}$$

with constants depending on the dimensions of the cube. We also introduce the following norms:

$$\|u\|_{\bar{r}} = \|u\| + \sum_i J\left[h^{-(r_i - r_i')}\|\Delta^2_{h,i} D_i^{r_i'} u\|\right], \tag{3}$$

$$\|u\|_s = \|u\| + \sup_{h_i > 0}(h_1 \cdots h_n)^{-s+s'}\|\Delta^2_{h_1,1}, \dots, \Delta^2_{h_n,n} D^s u\|, \tag{4}$$

where $\bar{r} = (r_1, \dots, r_n)$; r_i', s_i' are the greatest integers smaller, respectively, than r_i and s_i, $D_i^k = \partial^k/\partial x_i^k$, and J is an arbitrary functional of maximization type (see (6,7)). The spaces obtained by completing infinitely differentiable functions finite with respect to Ω in these norms will be denoted by H, $H^{\bar{r}}$, H^s. The analogous spaces obtained by replacing $\|\cdot\|$ by $\|\cdot\|_{L_2}$ will be denoted by L_2, $L_2^{\bar{r}}$, L_2^s. We denote by $A(M, \bar{r})$ and $B(M, s)$ the compact sets in H defined by the inequalities

$$\|u\|_{\bar{r}} \leq M; \qquad \|u\|_s \leq M, \tag{5}$$

and by $A_{L_2}(M, \bar{r})$, $B_{L_2}(M, s)$ the analogously defined compact sets in L_2. Our aim is to obtain upper estimates for the ε-entropy of $B(M, s)$ in H—$\psi(\varepsilon, M, s)$ —and lower estimates for the ε-entropy of $A(M, \bar{r})$ in H—$\Phi(\varepsilon, M, \bar{r})$.

Theorem. For any $\mu > 0$ and any compact sets $A(M, \bar{r})$ and $B(M, s)$ of the type described, the estimates

$$\Phi(\varepsilon, M, \bar{r}) > C_\mu (M\varepsilon^{-1})^{\rho - \mu}, \quad \rho = \sum_i r_i^{-1}, \tag{6}$$

$$\psi(\varepsilon, M, s) < C_\mu (M\varepsilon^{-1})^{1/s + \mu}. \tag{7}$$

hold.

Let us establish the necessary auxiliary propositions.

Lemma 1. For the compact sets $A_{L_2}(M, \bar{r})$ and $B_{L_2}(M, s)$, considered in L_2, the estimates (6) and (7) are valid.

The proof uses Parseval' s equality and elementary constructions in finite-dimensional spaces.

Lemma 2. For any integer N, the ε-entropy of $A(M, N\bar{r})$ in $H^{N\bar{r} - \bar{r}}$ does not exceed $C_\alpha \Phi(\varepsilon, M, \alpha\bar{r})$, where $\alpha < 1$ is arbitrarily close to unity.

The proof of this proposition is based on the fact that if $u \in A(M, N\bar{r})$, then $D_i^{(N-1)r_i} u \in A(M, \bar{r})$, and that the sum of the norms in H of such derivatives and of the function itself is a norm stronger than the norm in $H^{\alpha\bar{r}}$, with $\alpha < 1$ arbitrarily close to unity.

Lemma 3. If for some $\varepsilon > 0$ and $q > 0$

$$\Phi(\varepsilon, M, \bar{r}) < (M\varepsilon^{-1})^{q}, \tag{8}$$

then, for $\eta = \varepsilon^N$ and any $\delta > 0$,

$$\Phi(\eta, M, N\bar{r}) < C_\delta (M\eta^{-1})^{q/N+\delta}. \tag{9}$$

Construct a minimal ε-covering of $A(M, N\bar{r})$ in $H^{N\bar{r}-\bar{r}}$, the number of elements P_1 in which is estimated by Lemma 2. Then, for each element of the covering obtained, construct a minimal ε^2-covering in $H^{N\bar{r}-2\bar{r}}$ with number of elements P_2, and, finally, proceed to an ε^N-covering in H for all of $A(M, N\bar{r})$. The number of elements in the resulting covering will be equal to $P_1 \cdots P_N$, and the $\log P_i$ have the same order in ε. This gives the estimate (9).

Lemma 4. For any $\delta > 0$ there exists an N such that

$$\Phi(\eta, M, N\bar{r}) > C_\delta (M\eta^{-1})^{\rho/N-\delta}. \tag{10}$$

For the proof of (10) we use the inequalities, following from (1)-(2):

$$\|u\|_{L_2} \le C_1 \|D^l u\| \le C_2 \|D^l u\|_{N\bar{r}} \le C_3 \sum_i J \left[h^{-\varkappa_i} \left\| \Delta^2_{h,i} D_i^{nl+Nr_i-\varkappa_i} u \right\| \right] \le$$

$$\le C_4 \sum_i J \left[h^{-\varkappa_i} \left\| \Delta^2_{h,i} D^{2nl+Nr_i-\varkappa_i} u \right\|_{L_2} \right] \le Q,$$

which, for $\varkappa_i < 2$, allow the construction of ε-distinguishable sets for the compact sets $A(M, N\bar{r})$ in the metric H^l to be reduced to the construction of ε-distinguishable sets in L_2 in compact sets characterized by the given Q, and hence to estimate their ε-capacity with the aid of Lemma 1. In addition, the ε-capacity of $A(M, N\bar{r})$ in H^l is estimated from below through the ε^β-capacity of $A(M, N\bar{r})$ in H, with $\beta < 1$ close to unity. This follows from the estimates

$$\|D^l u\| \le C_1 \left(\sum_i \|D_i^{ln+1} u\| + \|u\| \right) \le C_2 \|u\|^\beta \|u\|_{N\bar{r}}^{1-\beta}, \tag{11}$$

where $(1-\beta)N \min r_k = ln + 1$, which are simple generalizations of the inequalities "sharp with respect to differential order" (8,9).

A comparison of Lemmas 3 and 4 shows that condition (8) of Lemma 3 is not satisfied for $q = \rho - \mu$, whatever $\mu > 0$ may be, starting with some sufficiently small ε. Consequently, estimate (6) holds.

Lemma 5. *If, for a given $\mu > 0$, estimate (7) is valid for some s, then the ε-entropy of $B(M,s)$ in H^{s_0}, for $s_0 < s$, does not exceed*

$$C\,(M\varepsilon^{-1})^{(1+\mu s)/(s-s_0)}.$$

This follows from the multiplicative estimate

$$\|u\|_{s_0} \le C\|u\|^{\frac{s-s_0}{s}}\|u\|_s^{\frac{s_0}{s}},$$

which shows that an ε-net of $B(M,s)$ in H will at the same time be a $C\varepsilon$-net of $B(M,s)$ in H^{s_0}.

Lemma 6. *For every $\mu > 0$ there is an s_0 such that, for $s > s_0$, estimate (7) holds.*

The assertion of the lemma follows from Lemma 1 and from the inequalities

$$\|u\| \le C_1\|u\|_{L_2^l} \le C_2\|u\|_{L_2^{s-1}} \le C_3\|u\|_{H^s},$$

which reduce the construction of ε-nets for $B(M,s)$ in H to their construction for $B_{L_2}(M',s)$ in L_2^l.

Lemmas 5 and 6 make it possible to prove estimate (7) for arbitrary s by induction on n. Let us choose an arbitrary μ and fix a sufficiently large integer $\tilde{s}_0$. Then for every $u \in B(M,s)$ we obtain the representation

$$u(x) = D^{\tilde{s}_0}v(x) + \sum_{k,i}^{\tilde{s}_0,n} g_k(x_i)\,r_{ki}(x),$$

where $g_k(z)$ are standard infinitely differentiable finite functions whose k-th moment is equal to one, while all the other moments from the zeroth to the $\tilde{s}_0$-th are equal to zero; $r_{k,i}(x)$ are functions of $n-1$ variables, independent of x_i, whose defining algorithm is easy to make unambiguous, and for $n = 1$ they are numbers. To estimate the contribution to the ε-entropy of the first term one may use Lemmas 5 and 6; moreover, in Lemma 5 one should take $s_0 > \tilde{s}_0$, close to $\tilde{s}_0$. The contributions of the remaining terms are taken into account by means of the induction hypothesis.

In conclusion, let us note that many compacta contain compacta of the form $A(M,\bar{r})$ and are contained in $B(M',s)$ with $s^{-1} = \rho$. In these cases estimates (6) and (7) describe well the asymptotic behavior of the ε-entropy. In particular, for $A(M,\bar{r})$ and $B(M,s)$ themselves we have

$$\lim_{\varepsilon\to 0}|\log\varepsilon|^{-1}\log\Phi(\varepsilon,M,\bar{r}) = \sum_i r_i^{-1}, \qquad \lim_{\varepsilon\to 0}|\log\varepsilon|^{-1}\log\psi(\varepsilon,M,s) = s^{-1}.$$

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