On the Quasi-Peanity of Functions

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Abstract

Full Text

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As in the note (1), let the symbol G denote the set of all functions from $P_{\aleph_0}$ that depend on no more than one variable. Further, a function which, together with the set G, forms a complete system in $P_{\aleph_0}$ will be called **quasi-Peanian**. In the note (1) one necessary and one sufficient condition for the quasi-Peanity of functions were formulated. In the present work a brief exposition is given of the complete solution of the question of the quasi-Peanity of functions. The basic result is the result establishing the existence in $P_{\aleph_0}$ of only two precomplete classes containing the set G. In connection with this fact it is of interest to note that in k-valued logics (P_k), for each $k \geq 2$, there is only one precomplete class containing the whole set G_k of functions depending on no more than one variable (2).

In k-valued logics ($k \geq 3$) the theorem (2) is known: the system

$$G_k \cup \{f(x_1, \dots, x_n)\}$$

is complete in P_k if and only if the function $f(x_1, \dots, x_n)$ essentially depends on not fewer than two variables and assumes all k values (is a so-called essential function). The analogy between quasi-Peanian and essential functions is evident. Denote by CS_k the subset of G_k consisting of all functions which omit at least one value. It is easy to see that CS_k is a closed class (with respect to the operation of superposition). It turns out that in P_k ($k \geq 3$) a function $f(x_1, \dots, x_n)$ is essential if and only if the system

$$CS_k \cup \{f(x_1, \dots, x_n)\}$$

is complete in P_k (2). It is not difficult to show that the direct transfer of this assertion to the case of countably valued logic does not give a positive result. The question arises whether one can narrow the set G to some proper subset M, which is a closed class and which, together with any quasi-Peanian function, forms a complete system in $P_{\aleph_0}$. A closed class M from G (respectively from G_k, $k \geq 3$) will be called a ξ-class if, whatever the quasi-Peanian (respectively essential) function f may be, the system $M \cup \{f\}$ is complete in $P_{\aleph_0}$ (respectively in P_k). Denote by $S_{E^{\aleph_0}}$ the set of all many-valued functions from G that assume every value from the set

$$E^{\aleph_0} = \{0, 1, 2, \dots\}.$$

In the present note a necessary and sufficient condition is formulated for a set containing $S_{E^{\aleph_0}}$ to be a ξ-class. In connection with the indicated result let us recall that in P_k, for $k \geq 5$, every closed class containing the set S_k of all many-valued functions is a ξ-class ([3]). In P_3, as is not difficult to show, only the set G_3 is such a ξ-class. In P_4, as we have established, of the six closed classes (from G_4) containing the subset S_4, only three are ξ-classes.

1°. Let us introduce the necessary notation and definitions. By $G^{(1)}$ we denote the set of all functions from $P_{\aleph_0}$ that depend on one variable; $C(m)$ will denote the subset of functions from the set $G^{(1)}$ that assume exactly m different values $(1 \leq m < \aleph_0)$;

$$C_\omega = \bigcup_{m=1}^{\infty} C(m).$$

Let $f(x, y)$ be an arbitrary function from $P_{\aleph_0}$ depending on two variables. If either $f(x, k) \in C_\omega$ for every k, or $f(l, y) \in C_\omega$ for every–

some l (k and l belong to the set $E^{\aleph_0}$), then the function $f(x, y)$ will be called a **function of the first kind**. The set of all functions of the first kind from $P_{\aleph_0}$ will be denoted by R_1.

We shall say that a function $f(x, y)$ from $P_{\aleph_0}$ **has a Peano angle in the first** (respectively, second) **variable** if the function $f(x + y, y)$ (respectively, $f(x, x + y)$) is Peano ([1]). Further, we shall say that a function $f(x, y)$ from $P_{\aleph_0}$ **has a quasi-Peano angle in the first** (respectively, second) **variable** if there exist functions $g_1(x)$ and $g_2(x)$ from $G^{(1)}$ such that the function $\varphi(x, y) = f(g_1(x), g_2(y))$ has a Peano angle in the first (respectively, second) variable. It is clear that in this definition $g_1(x)$ and $g_2(x)$ are one-to-one functions. It is easy to show that in the above definition it suffices to restrict oneself to the subset of strictly monotone functions from $G^{(1)}$. We shall call $f(x, y)$ a **function of the second kind** if it does not have a quasi-Peano angle in either of the variables. The set of all functions of the second kind from $P_{\aleph_0}$ will be denoted by R_2.

Consider the set of variables $\{u_1, u_2\}$. Denote by $H^{(1)}$ the set of all functions from $P_{\aleph_0}$ depending on one of the two variables u_1 and u_2. Denote by $H^{(2)}$ the set of all functions from $P_{\aleph_0}$ depending on both variables u_1 and u_2. $G^{(0)}$ will denote the set of all functions from $P_{\aleph_0}$ depending on zero variables. By the symbol F_1 we denote the subset of $H^{(2)}$ consisting of all functions of the first kind, and by F_2–of all functions of the second kind. $T_i = G^{(0)} \cup H^{(1)} \cup F_i$, $i = 1, 2$. $\mathfrak{P}(T_i)$ is the closure class ([2]) of the set T_i $(i = 1, 2)$.

2°. Let us formulate a criterion for the quasi-Peano property of functions.

Theorem 1. *A function $f(x_1, \dots, x_n)$ $(n \geqslant 2)$ is quasi-Peano if and only if it does not belong to the set*

$$\mathfrak{P}(T_1) \cup \mathfrak{P}(T_2).$$

In the proof of this theorem several auxiliary assertions are used, the principal ones being the following.

Lemma 1. *Let a function $f(x_1, \dots, x_n)$ $(n \geqslant 2)$ not preserve* ([2]) *the set T_1. Then it does not preserve it also on the set $G^{(0)} \cup H^{(1)}$, i.e. there exist functions $g_1, \dots, g_n$ from $G^{(0)} \cup H^{(1)}$, upon substituting which in place of the variables in the function f one obtains a function (from $H^{(2)}$) not belonging to T_1.*

Remark. By I_0 we denote the set of all functions of large range ([4]) from $G^{(1)}$. By the symbol I_1 we denote the set of all functions from $G^{(1)}$ satisfying the following two conditions: a) the function omits $\aleph_0$ values from the set $E^{\aleph_0}$; b) each level set ([4]) of the function (with respect to the number it assumes) is infinite. Using this notation, we formulate two assertions:

1. *If a function $f(x_1, \dots, x_n)$ $(n \geqslant 2)$ does not preserve the set T_1, then it does not preserve it also on the set $H^{(1)} \cap I_0$.*
2. *If a function $f(x_1, \dots, x_n)$ $(n \geqslant 2)$ does not preserve the set T_1, then it does not preserve it also on the set $H^{(1)} \cap I_1$.*

Lemma 2. *If a function $f(x, y)$ does not have a quasi-Peano angle in the first variable, then there exist strictly monotone functions $g_1(x)$ and $g_2(x)$ such that the function $\varphi(x, y) = f(g_1(x), g_2(y))$ satisfies at least one of the two conditions (the first and the second conditions of degeneracy):*

$$1.\ \varphi(x + y, y) = \varphi(y, y). \qquad 2.\ \varphi(x + y, y) = \varphi(x + y, 0).$$

Lemma 3. *If a function $f(x_1, \dots, x_n)$ $(n \geqslant 2)$ does not preserve the set T_2, then it does not preserve it also on the set $G^{(0)} \cup H^{(1)}$.*

Remark. In the formulation of this lemma, the set $G^{(0)} \cup H^{(i)}$ may be replaced by either of the two sets $H^{(1)} \cap I_0$ and $H^{(1)} \cap I_1$.

Lemma 4. If a function $f(x, y)$ has a quasi-Peano node with respect to the first (respectively, the second) variable, then there exist strictly monotone functions $g_1(x)$ and $g_2(x)$ and a function $g(x)$ such that the function $\varphi(x, y) = g(f(g_1(x), g_2(y)))$ has a Peano node with respect to the first (respectively, the second) variable and $\varphi(x, x + y + 1) \equiv 0$ (respectively, $\varphi(x + y + 1, y) \equiv 0$).

3°. **Theorem 2.** If the function $f(x_1, \dots, x_n)$ $(n \geq 2)$ is quasi-Peano, then it has quasi-Peano order ([1]) not exceeding 2.

Theorem 3. There exist only two precomplete classes containing the set of functions G, namely, $\mathfrak{P}(T_1)$ and $\mathfrak{P}(T_2)$.

Theorem 4. A function $f(x, y)$ is quasi-Peano of the first order if and only if there exist functions $g_1(x)$ and $g_2(x)$ such that the function $\varphi(x, y) = f(g_1(x), g_2(y))$, when the first variable is fixed, is a many-valued function of the second variable, and when the second is fixed, is a many-valued function of the first.

Theorem 5. If the function $f(x, y)$ does not belong to the set $R_1 \cup R_2$, then it is quasi-Peano.

4°. By π_0 we shall denote the operation of substituting for the variables in a function $f(x_1, \dots, x_n) \in P_{\aleph_0}$ functions from the set $G^{(1)}$, followed by a renaming of the variables.

Theorem 6. Let $f(x_1, \dots, x_n)$ be a quasi-Peano function. If $n \geq 5$, then by means of the operation π_0 one can obtain from it a quasi-Peano function of a smaller number of variables. If, however, $n = 3$ or $n = 4$, then obtaining from the function f, by means of the operation π_0, a quasi-Peano function depending on a smaller number of variables is not always possible.

Consider a quasi-Peano function $f(x_1, \dots, x_n)$. Denote by $K(f, n)$ the minimal number of functions from the set $G^{(1)}$ that must be used in order, from the function f by means of the operation π_0, to construct functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ not belonging respectively to the sets R_1 and R_2. Let $K(n) = \max_f K(f, n)$, where the maximum is taken over all quasi-Peano functions of n variables.

Theorem 7. $K(n) = n - 2$.

5°. Consider several closed classes in the set G: Q_5 consists of the set $G^{(0)}$ and all such functions from the set $G^{(1)}$ which: 1) either assume no more than a finite number of values from $E^{\aleph_0}$, 2) or have an infinite complement to the union of all singleton level sets; Q_6 consists of the set $G^{(0)}$ and all such functions from the set $G^{(1)}$ which: 1) either have repetitions, 2) or do not assume a single value from $E^{\aleph_0}$; $S_0 = I_0 \cup G^{(0)} \cup S_{E^{\aleph_0}} \cup C(1) \cup (C(2) \cap I_1)$.

Theorem 8. Let Q be a closed class in G containing the set $S_{E^{\aleph_0}}$ ($G \supseteq Q \supset S_{E^{\aleph_0}}$). In order that Q be a ξ-class, it is necessary and sufficient that the condition $Q \supseteq S_0$ be satisfied.

Theorem 9. Let Q be a ξ-class containing the set $S_{E^{\aleph_0}}$ and different from the whole set G. Then either $Q_5 \supseteq Q$, or $Q_6 \supseteq Q$.

Remark. Everywhere in this item the conjunction "or" is inseparable.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.