MATHEMATICS

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Abstract

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ON EXPANSIONS IN EIGENFUNCTIONS OF SYSTEMS WITH A SUMMABLE POTENTIAL

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1. It is known that every boundary-value problem for a self-adjoint system of differential equations on the half-line

$$-y'' + P(x)y = \lambda y, \qquad 0 \le x < \infty; \tag{1}$$

$$y'(0) - Hy(0) = 0, \tag{2}$$

where $y = (y_1(x), \dots, y_n(x))$ is a column vector; $H = H^*$ and $P(x) = P^*(x)$ are Hermitian matrices of order n; $P(x)$ is locally summable with respect to x, gives rise to inversion formulas in the space of vector-functions $L_n^2(0, \infty)$:

$$F(\lambda) = \int_0^\infty \omega^*(x,\lambda) f(x)\, dx, \qquad f(x) = \int_{-\infty}^\infty \omega(x,\lambda)\,[d\rho(\lambda)]F(\lambda). \tag{3}$$

Here $\omega(x,\lambda)$ is the matrix solution of problem (1), (2), $\rho(\lambda)$ is a matrix non-decreasing distribution function—the spectral function of the problem, and the integrals converge in the mean-square sense. In the case of a potential $P(x)$ with summable norm

$$\int_0^\infty \|P(x)\|\, dx < \infty \tag{4}$$

the spectral function of problem (1), (2) is unique for the given normalization of the eigenfunctions $\omega(x,\lambda)$ ([1]).

Under condition (4), below the concrete form of the spectral function $\rho(\lambda)$ (and, consequently, of the expansion formulas (3)) will be established by the known method of passage to the limit from a finite interval ([2], [3]), and some generalizations of problem (1), (2), (4) will also be considered. We note that for the case

$$\int_0^\infty x\|P(x)\,dx\| < \infty$$

with the boundary condition $y(0) = 0$, expansion formulas were obtained by another method in ([4]). This case is covered by our considerations.

2. Denote by $E(x, s)$ the matrix solution of equation (1) with summable potential (4), having the asymptotic form

$$E(x, s) = e^{isx}[I + o(1)], \qquad x \to \infty, \qquad \operatorname{Im} s \geq 0 \tag{5}$$

($\lambda = s^2$, I is the identity matrix, $o(1)$ is a matrix with infinitesimal norm). Such a solution was constructed in ([5]). For real $s \neq 0$, the functions $E(x, s)$ and $E(x, -s)$ form a fundamental system of solutions. To construct the inversion formulas we use solutions of equation (1) of the form

$$U(x, s) = E(x, s) - E(x, -s)A(s), \qquad s^2 > 0, \tag{6}$$

satisfying condition (2), in view of which

$$A(s) = Z^{-1}(0, -s)Z(0, s), \tag{7}$$

where

$$Z(x, s) = E'(x, s) - HE(x, s). \tag{8}$$

The existence of $A(s)$ is ensured by the following lemma.

Lemma 1. *For real $s \neq 0$ and any $x \geq 0$, the matrix $Z(x, s)$ has an inverse. The same is true for $E(x, s)$.*

Proof. It is known ([4]) that, for any real s,

$$E^*(x, s)E'(x, s) - E^{*'}(x, s)E(x, s) = -2isI. \tag{9}$$

Therefore, for any vector $\mathbf{a}$, we obtain

$$\mathbf{a}^*[Z^{*'}(x, s)E(x, s) - E^*(x, s)Z'(x, s)]\mathbf{a} = -2is|\mathbf{a}|^2,$$

and hence, if $Z\mathbf{a} = 0$, then $\mathbf{a} = 0$. For $Z(x, s)$ the lemma is proved, and the nonsingularity of $E(x, s)$ follows from (9) and was shown in ([4]).

Lemma 2. *For $\operatorname{Im} s = 0$, $x \geq 0$, the identities hold:*

$$E^*(x,s)E^*(x,s) = E(x,-s)E^*(x,-s),$$

$$E'(x,s)E^{*'}(x,s) = E'(x,-s)E^{*'}(x,-s), \tag{10}$$

$$E'(x,s)E^*(x,s) - E'(x,-s)E^*(x,-s) = 2isI.$$

The first of these equalities was obtained in ([4]) (for $x = 0$); the others are proved analogously.

Lemma 3. *The matrix $A(s)$ (7), (8) is unitary.*

Proof. From the identities (10) it follows that $Z(x,s)Z^*(x,s) = Z(x,-s)Z^*(x,-s)$, and therefore $A^*(s) = A^{-1}(s)$, as was required.

Let $\xi_1(s), \dots, \xi_n(s)$ be the eigenvalues of the matrix $A(s)$, and let $\mathbf{e}_1(s), \dots, \mathbf{e}_n(s)$ be the corresponding orthonormal system of eigenvectors. All these quantities depend continuously on $s \neq 0$, and

$$A(s)\mathbf{e}_j(s) = \xi_j(s)\mathbf{e}_j(s), \qquad |\xi_j(s)| = |\mathbf{e}_j(s)| = 1, \qquad j = 1, \dots, n. \tag{11}$$

Consider the auxiliary boundary-value problem (1), (2) on the interval $(0, b)$, imposing at the right end of the interval the condition $\mathbf{y}(b) = 0$. In the works ([5]), under the additional assumption that the matrix $A(s)$ has a complete set of orthogonal eigenvectors (the unitarity of $A(s)$ had not been discovered), asymptotics were obtained for the eigenvalues and eigenfunctions of this problem as $b \to \infty$. Lemma 3 justifies the assumptions made in ([5]) and the asymptotics obtained there. By virtue of these asymptotics, we have, slightly refining the formulation, that the positive eigenvalues of the auxiliary problem form n sequences $\lambda_{jk} = s_{jk}^2(b)$, $j = 1, \dots, n$, $k \to \infty$, uniformly for all s_{jk} from any interval $0 < \varepsilon \leq s_{jk} \leq N$, and

$$s_{j,k+1} - s_{jk} = \frac{\pi}{b} + o\left(\frac{1}{b}\right), \qquad s_{jk} - s_{pk} = O\left(\frac{1}{b}\right). \tag{12}$$

The corresponding eigenvector-functions have the form $(0 \leq x \leq b)$

$$\mathbf{y}_{jk}(x) = U(x, s_{jk})\mathbf{e}_j(s_k) + o(1), \qquad s_k = \frac{k\pi}{b}. \tag{13}$$

Therefore, by virtue of (5), (6), and (11),

$$\int_0^b |\mathbf{y}_{jk}(x)|^2\,dx = 2b[1 + o(1)]. \tag{14}$$

Now let us obtain $\rho(\lambda)$ as the limit, as $b \to \infty$, of the spectral function $\rho_b(\lambda)$ of the auxiliary problem on the interval $(0, b)$. To this end, note that

$$\sum_{j=1}^{n} \mathbf{e}_j(s)\mathbf{e}_j^*(s) = I,$$

where $\mathbf{ac}^*$ denotes the operator $(\dots, \mathbf{c})\mathbf{a}$. For $\lambda > \mu > 0$, we have, by virtue of (12), (13), (14):

$$\rho_b(\lambda) - \rho_b(\mu) = \sum_{\mu<\lambda_{jk}<\lambda} \mathbf{e}_j(s_k)\mathbf{e}_j^*(s_k)\frac{1}{2b} + o(1) =$$

$$= \frac{1}{2\pi} \sum_{\mu<s_k^2<\lambda} \left\{ \sum_{j=1}^{n} \mathbf{e}_j(s_k)\mathbf{e}_j^*(s_k) \right\} (s_{k+1} - s_k) + o(1) \to I \frac{1}{2\pi} \int_{\mu}^{\lambda} d\sqrt{\lambda}.$$

Thus, it has been proved:

Theorem 1. *The spectral function $\rho(\lambda)$ of the self-adjoint problem (1), (2), (4) is determined for $\lambda > 0$ by the equality*

$$\rho(\lambda) - \rho(+0) = \frac{1}{2\pi} I\sqrt{\lambda}, \tag{15}$$

if the eigenfunctions are normalized by condition (6).

Corollary. *If the eigenfunctions $\omega(x, \lambda)$ of problem (1), (2), (4) are normalized by the condition $\omega(0, \lambda) = I$, then the spectral function $\tilde{\rho}(\lambda)$ for $\lambda > 0$ has the form*

$$\tilde{\rho}(\lambda) = \tilde{\rho}(+0) + \frac{2}{\pi} \int_0^{\sqrt{\lambda}} [Z(0, s)Z^*(0, s)]^{-1} s^2 \, ds. \tag{16}$$

For $\lambda < 0$, as is known ([4],[5]), the spectrum is purely discrete, bounded from below, and can accumulate only at the point $\lambda = 0$. Let us add that $\lambda = 0$ also may turn out to be an eigenvalue.** (For $x = \|P(x)\| \in L(0, \infty)$ this possibility is excluded, and the number of eigenvalues is finite ([4]).)

3. Consider the self-adjoint problem for equation (1) with boundary condition

$$\cos A \, y'(0) - \sin A \, y(0) = 0, \tag{17}$$

where A is an arbitrary Hermitian matrix. Since both matrices $\cos A$ and $\sin A$ may be singular, condition (17) does not reduce, in general, to condition (2) and has no analogue in the scalar case. At the same time problem (1), (17), (4) has solutions $U(x, s)$ of the form (6), where $A(s)$ is a unitary matrix.***

Theorem 2. *Under normalization of the eigenfunctions by condition (6), the boundary-value problem (1), (17), (4) has spectral function (15), and under normalization by the initial values* $\omega(0, \lambda) = \cos A$, $\omega'(0, \lambda) = \sin A$ *it has spectral function (16), where*

$$Z(x, s) = \cos A\, E'(x, s) - \sin A\, E(x, s) \tag{18}$$

is a nonsingular matrix. The negative spectrum is discrete and bounded from below. In the case of the boundary condition $y(0) = 0$, *the restriction (4) on the potential may be replaced by the weaker one*

$$\int_0^1 x\|P(x)\|\, dx < \infty, \qquad \int_0^\infty \|P(x)\|\, dx < \infty.$$

4. Let now (1) be an infinite system of differential equations; $y(x)$ a vector-function with values in a separable Hilbert space $\mathfrak{H}$; $P(x)$ for each x and A in (17) bounded self-adjoint operators in $\mathfrak{H}$; $P(x)$ depends continuously on x. The corresponding inversion formulas generate an operator-valued distribution function $\rho(\lambda)$, which can be obtained, for example, by the limiting-transition method from problems on a finite interval ([6]), Ch. III, § 7. If the norm of $P(x)$ as an operator in $\mathfrak{H}$ satisfies condition (4), then the following assertions are valid:

Lemma 4. *The operator* $A(s)$, (6), (7), (18), *exists and is unitary.*

Theorem 3. *The spectrum of problem (1), (17), (4) with operator coefficients is purely continuous for* $\lambda > 0$, *and for the spectral function* $\rho(\lambda)$ *one has*

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* We note that in ([5]), under the conditions of Theorem 1, an erroneous expression was obtained for $\rho(\lambda)$, which gives an incorrect result even for $H = P(x) \equiv 0$, because, in particular, $\rho'(\lambda)$ turns out in ([5]) to be a singular matrix of rank 1.

** For example, the equation $y'' - 2(x+1)^{-2}y + \lambda y = 0$ with the boundary condition $y'(0) + y(0) = 0$ has, for $\lambda = 0$, the eigenfunction $y_0(x) = (x+1)^{-1} \in L^2(0, \infty)$.

*** Under the condition $y(0) = 0$, the unitarity of $A(s)$ is shown in ([4]).

formula (15) is valid under the normalization (6). For $\lambda < 0$ the spectrum is bounded below, but, in contrast to the case of finite systems, it may be both discrete and continuous.

Example. If $P(x) \equiv 0$, then the negative spectrum of problem (1), (2) consists of those λ for which $-\sqrt{|\lambda|}$ is a point of the spectrum of the operator H. In this case, points of the discrete or continuous spectrum of the operator H correspond, respectively, to points of the discrete or continuous spectrum of problem (1), (2).

An analogous phenomenon of "smearing" of the discrete spectrum of a problem on a finite interval when passing to infinite systems of equations was noted in (6).* It is connected with the noncompactness of the unit sphere in $\mathfrak{H}$.

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* Taking this opportunity, we note that the spectrum of an operator boundary-value problem on a finite interval $(0, b)$ is contained in bounded intervals $|\lambda - n^2\pi^2/b^2| <$ const, whose images on the s-axis, $s = \sqrt{\lambda}$ (but not the intervals themselves on the λ-axis, as stated in (6), p. 337), contract as $\lambda \to +\infty$ to the points $s_n = n\pi/b$.

Note: Figure translations are in progress. See original paper for figures.