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## Abstract

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# NONLINEAR THEORY OF THE DRIFT-CYCLOTRON INSTABILITY OF A NON-ISOTHERMAL PLASMA

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Recently it has been shown that various types of instabilities can develop in a plasma, the cause of which is the spatial inhomogeneity of the particle density or temperature ([1]). These instabilities are associated with oscillations whose frequencies are small or comparable with the ion cyclotron frequency. The case studied in greatest detail is that of low-frequency oscillations corresponding to the so-called drift instability. The analysis of drift instability carried out up to the present time has not only made it possible to clarify the form of the dispersion equation and the relations following from it, which is the task of the linear approximation of the theory of stability, but has also made it possible to determine, at least qualitatively, what consequences this instability leads to ([2,3]).

Other types of instability of an inhomogeneous plasma, in particular cyclotron instability, have apparently been investigated only on the basis of linearized equations ([4]). In the present work equations are obtained and analyzed which take nonlinear effects into account, as applied to the problem of the cyclotron instability of an inhomogeneous nonisothermal ($T_e \gg T_i$) plasma. A plasma with $T_e \gg T_i$ is quite often realized experimentally (for example, in ([5])). At the same time this case is the simplest from the standpoint of theoretical investigation, since the growth rate of the cyclotron oscillations is then comparatively small, the coupling between the oscillations is weak, and the nonlinear interaction between them can be described by means of the equations of the theory of weak turbulence ([2,3]).

The initial equations for our problem can be obtained by the method set forth in ([2]). For this purpose one must solve the Vlasov equation for the plasma charges with accuracy up to terms of third order inclusive in the wave amplitude. By the method of successive approximations one can find that the part of the distribution function of each species of charges which oscillates in time has the form:

$$f_{\mathbf{k}\omega} = f^{\mathrm{I}}_{\mathbf{k}\omega} + f^{\mathrm{II}}_{\mathbf{k}\omega} + f^{\mathrm{III}}_{\mathbf{k}\omega}, \tag{1}$$

where

$$f^{\mathrm{I}}_{\mathbf{k}\omega} = -\frac{e\varphi_{\mathbf{k}\omega}}{T} f_0 \left\{ 1 - (\omega - \omega^*_k) \sum_{n=-\infty}^{+\infty} \frac{J^*_n \exp\left[i\xi_k \sin(\alpha - \psi_k) - in(\alpha - \psi_k)\right]}{\omega - n\omega_B - k_z v_z} \right\}; \tag{2}$$

$$f^{\mathrm{II}}_{\mathbf{k}\omega} = \int L_{\mathbf{k}\omega,\mathbf{k}'\omega'} \left( \varphi_{\mathbf{k}'\omega'} \varphi_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} - \left\langle \varphi_{\mathbf{k}'\omega'} \varphi_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} \right\rangle \right) d\mathbf{k}'\, d\omega'; \tag{3}$$

$$\begin{aligned} f^{\mathrm{III}}_{\mathbf{k}\omega} = \int M_{\mathbf{k}\omega,\mathbf{k}'\omega',\mathbf{k}''\omega''} \varphi_{\mathbf{k}''\omega''} \Big( & \varphi_{\mathbf{k}'\omega'} \varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\,\omega-\omega'-\omega''} \\ & - \left\langle \varphi_{\mathbf{k}'\omega'} \varphi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}'',\,\omega-\omega'-\omega''} \right\rangle \Big) d\mathbf{k}'\, d\omega'\, d\mathbf{k}''\, d\omega''. \end{aligned} \tag{4}$$

$$\begin{gathered} L_{\mathbf{k}\omega,\mathbf{k}'\omega'} = -\frac{ie^2 f_0 (\tilde\omega - \tilde\omega')[\mathbf{k},\mathbf{k}']_z}{m\omega_B T} \times \\ \times \sum_{n,n'} \frac{J^{\mathbf{k}-\mathbf{k}'}_n J^{\mathbf{k}'}_{n'} \exp\{i\xi_{\mathbf{k}} \sin(\alpha - \psi_{\mathbf{k}}) - i(n+n')\alpha + in\psi_{\mathbf{k}-\mathbf{k}'} + in'\psi_{\mathbf{k}'}\}}{[\omega - \omega' - (k_z - k'_z)v_z - n\omega_B][\omega - k_z v_z - (n+n')\omega_B]}; \end{gathered} \tag{5}$$

$$\begin{gathered} M_{\mathbf{k},\omega;\mathbf{k}'\omega';\mathbf{k}''\omega''} = -\frac{e^3 f_0 (\tilde\omega - \tilde\omega' - \tilde\omega'')[\mathbf{k},\mathbf{k}'']_z [\mathbf{k}-\mathbf{k}'',\mathbf{k}']_z}{m^2 \omega_B^2 T} \times \\ \times \sum_{n,n',n''} \Big\{ \{ J^{\mathbf{k}''}_{n''} J^{\mathbf{k}-\mathbf{k}'-\mathbf{k}''}_n J^{\mathbf{k}'}_{n'} \exp[\, i\xi_{\mathbf{k}} \sin(\alpha - \psi_{\mathbf{k}}) - i\alpha(n+n'+n'') + \\ + in\psi_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} + in'\psi_{\mathbf{k}'} + in''\psi_{\mathbf{k}''}]\} \{ [\omega - \omega' - \omega'' - v_z(k_z - k'_z - k''_z) - \\ - n\omega_B][\omega - \omega'' - v_z(k_z - k''_z) - (n+n')\omega_B][\omega - k_z v_z - (n+n'+n'')\omega_B] \}^{-1} \Big\}; \end{gathered} \tag{6}$$

$$\xi_{\mathbf{k}} = \frac{k_\perp v_\perp}{\omega_B}, \qquad \psi_{\mathbf{k}} = \operatorname{arc\,tg}\left(\frac{k_y}{k_x}\right), \qquad k_\perp = (k_x^2 + k_y^2)^{1/2}, \qquad J^{\mathbf{k}}_n = J_n(\xi_{\mathbf{k}}),$$

$$\alpha = \operatorname{arc\,tg}\left(\frac{v_y}{v_x}\right), \qquad v_\perp = (v_x^2 + v_y^2)^{1/2}, \qquad \varkappa = \frac{\partial \ln N}{\partial y},$$

$$\omega^*_{\mathbf{k}} = k_x v^*, \qquad v^* = -\frac{\varkappa T}{m\omega_B}, \qquad \tilde\omega = \omega - \omega^*_{\mathbf{k}}, \qquad \omega_B = \frac{eB}{mc},$$

$$f_0 = \left(\frac{m}{2\pi T}\right)^{3/2} N \exp\left[-\frac{mv^2}{2T}\right], \qquad \nabla T = 0, \qquad \nabla N \neq 0,$$

$J_n$ is the Bessel function. The resonant denominators are written so that, for the proper bypassing of the poles, $+i\Delta$ ($\Delta > 0$) should be added to them. In calculating $f_{\mathbf{k}\omega}$ we assumed the oscillations to be potential, $\mathbf{E}_{\mathbf{k}\omega} = -i\mathbf{k}\varphi_{\mathbf{k}\omega}$, which is admissible if $\varkappa^2 \frac{T_i}{m_i\omega_{B_i}^2} < \frac{m_e}{m_i\beta_e}$; $\beta_e = \frac{8\pi N T_e}{B^2}$.

Using (1)–(6) we find the charge density, which we substitute into Poisson' s equation. Then, in the standard way, we construct the chain of equations for the correlation functions, truncating it by replacing the fourth-order correlation by a product of pair correlations. As a result we arrive at an equation containing only the double correlation function $I_{\mathbf{k}\omega} = \delta^{-1}(\mathbf{k}+\mathbf{k}')\delta^{-1}(\omega+\omega')\langle\varphi_{\mathbf{k}\omega}\varphi_{\mathbf{k}'\omega'}\rangle$:

$$\mathbf{k}^2[\varepsilon_0(\mathbf{k},\omega)+\varepsilon_1(\mathbf{k},\omega)+\varepsilon_2(\mathbf{k},\omega)]I_{\mathbf{k}\omega} = F_{\mathbf{k}\omega}. \tag{7}$$

Here $\varepsilon_0(\mathbf{k},\omega)$ is the scalar dielectric permittivity of the plasma in the linear approximation[1]

$$\varepsilon_0(\mathbf{k},\omega) = \frac{4\pi e^2 N}{k^2 T_i}\left\{1 + \frac{i\sqrt{\pi}(\omega - k_x v_i^*)}{|k_z| v_{T_i}} \sum_{n=-\infty}^{+\infty} I_n(z) e^{-z} W\left(\frac{\omega - n\omega_{B_i}}{|k_z| v_{T_i}}\right) + \right.$$
$$\left. + \frac{T_i}{T_e}\left[1 + \frac{i\sqrt{\pi}(\omega - k_x v_e^*)}{|k_z| v_{T_e}} W\left(\frac{\omega}{|k_z| v_{T_e}}\right)\right]\right\}; \tag{8}$$

$$z = \frac{k_\perp^2 T_i}{m_i \omega_{B_i}^2}, \qquad W(x) = e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\right), \qquad v_T = \sqrt{\frac{2T}{m}}.$$

$\varepsilon_1(\mathbf{k},\omega)$ and $\varepsilon_2(\mathbf{k},\omega)$ have the form:

$$\varepsilon_1(\mathbf{k},\omega) = -\frac{4\pi e^2}{\mathbf{k}^2}\int I_{\mathbf{k}'\omega'}\left(\overline{M}_{\mathbf{k}\omega,\mathbf{k}\omega,\mathbf{k}'\omega'} + \overline{M}_{\mathbf{k}\omega;-\mathbf{k}',-\omega',\mathbf{k}'\omega'}\right) d\mathbf{k}'\,d\omega'; \tag{9}$$

$$\varepsilon_2(\mathbf{k},\omega) = -\frac{(4\pi e)^2}{\mathbf{k}^2}\int \frac{I_{\mathbf{k}'\omega'}\,d\mathbf{k}'\,d\omega'}{(\mathbf{k}-\mathbf{k}')^2\varepsilon_0(\mathbf{k}-\mathbf{k}',\omega-\omega')}\left(\overline{L}_{\mathbf{k}-\mathbf{k}',\omega-\omega';-\mathbf{k}',-\omega'} + \right.$$
$$\left. + \overline{L}_{\mathbf{k}-\mathbf{k}',\omega-\omega';\mathbf{k},\omega}\right) \times \left(\overline{L}_{\mathbf{k}\omega;\mathbf{k}-\mathbf{k}',\omega-\omega'} + \overline{L}_{\mathbf{k}\omega,\mathbf{k}'\omega'}\right), \tag{10}$$

and the quantity $F_{\mathbf{k}\omega}$ is equal to

$$F_{\mathbf{k}\omega} = \frac{1}{2}\frac{(4\pi e)^2}{\mathbf{k}^2\varepsilon_0^*(\mathbf{k},\omega)}\int I_{\mathbf{k}'\omega'} I_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\left|\overline{L}_{\mathbf{k}\omega;\mathbf{k}'\omega'} + \overline{L}_{\mathbf{k}\omega;\mathbf{k}-\mathbf{k}',\omega-\omega'}\right|^2 d\mathbf{k}'\,d\omega'. \tag{11}$$

A bar over the quantities $L$ and $M$ denotes integration over velocities; $\varepsilon_0^*$ is the complex conjugate of $\varepsilon_0$.

In the linear approximation it follows from (8) that $I_{k\omega} = I_k \delta(\omega - \omega_k)$, where $\omega_k$ satisfies the condition $\varepsilon_0(k, \omega_k) = 0$. If $\varepsilon_0$ from (8) is substituted into the latter, then for

$$\frac{k_z v_{T_e}}{n\omega_{B_i}} \gg 1, \qquad \left|\frac{\omega - n\omega_{B_i}}{k_z v_{T_i}}\right| \gg 1, \qquad z \gg 1, \qquad T_e \gg T_i$$

we obtain[4]

$$\operatorname{Re}\omega_k = n\omega_{B_i}\left(1 + \frac{1 - k_x v_i^* / \operatorname{Re}\omega_k}{\sqrt{2\pi z}}\right), \tag{12}$$

$$\operatorname{Im}\omega_k = \gamma_k^{(0)} = \frac{1}{\sqrt{2z}}\,\frac{T_i}{T_e}\,\frac{\omega_k}{|k_z|v_{T_e}} \exp\left[-\frac{\omega_k^2}{(k_z v_{T_e})^2}\right](k_x v_e^* - \omega_k). \tag{13}$$

It is seen from the last equation that the cyclotron oscillations are unstable, $\gamma_k^{(0)} > 0$, if the electron drift velocity exceeds the phase velocity of the wave, $v_e^* > \omega/k_x$. (For this reason we call the instability under consideration drift-cyclotron.) The intensity of the oscillations will grow until the terms quadratic in $I_{k\omega}$ in equation (7), describing the interaction between the oscillations, become substantial. In our problem the most important role in the establishment of the stationary state is played by the effect of nonlinear damping on ions (a process of the type $\omega \rightleftarrows \omega'$), associated with $\varepsilon_1(\mathbf{k}, \omega)$ and partly with $\varepsilon_2(\mathbf{k}, \omega)$, and not the effect of decay and coalescence of oscillations ($\omega \rightleftarrows \omega' + \omega''$), associated with $F_{k\omega}$ and the corresponding part of $\varepsilon_2(\mathbf{k}, \omega)$. (For oscillations whose frequencies lie near one and the same cyclotron harmonic, the decay condition $\omega = \omega' + \omega''$ is obviously not satisfied, while decay processes involving two harmonics lead, as additional analysis shows, to small corrections of order $(\omega - n\omega_B)/n\omega_B$.) Taking these circumstances into account, carrying out the integration over velocities in $\overline{M}$ and $\overline{L}$, and making the same assumptions regarding the frequencies and phase velocities as in deriving (12), (13), one can transform (7) to the form

$$\frac{dI_k}{dt} = 2I_k(\gamma_k^{(0)} - \gamma_k^{(1)}), \tag{14}$$

where

$$\gamma_k^1 = \frac{2e^2}{m_i^2 \omega_{B_i}^2}\,\frac{k_\perp^6}{(n\omega_{B_i})^2}\,k_\perp |v_i^*| \ln z \int \sin^2(\psi - \psi')(\cos\psi - \cos\psi')\times$$
$$\times I(k_\perp, \psi', k_z')\, d\psi'\, dk_z'. \tag{15}$$

Here, as in (12), (13), it should be borne in mind that $\cos\psi, \cos\psi' > 0$, since for $\cos\psi < 0$ the unstable oscillations correspond to $\omega = -n\omega_B$.

By integrating (14) over $k_z$ and $\psi$, one can obtain an approximate equation for the function

$$I(k_\perp) = \int I(k_\perp, k_z, \psi) \cos\psi\, d\psi\, dk_z$$

$$\frac{\partial I_{k_\perp}}{\partial t} \simeq 2\varkappa\sqrt{\frac{T_i}{m_i}}\,I_{k_\perp}\left\{1-\frac{e^2k_\perp^7\rho_i^4 I_{k_\perp}}{(m_i\omega_{B_i}n)^2}\right\}, \qquad k_\perp > k_0 = \frac{n\omega_{B_i}}{v_e^*}, \qquad \rho_i = \sqrt{\frac{T_i}{m_i\omega_{B_i}^2}}. \tag{16}$$

In the stationary state $\partial/\partial t = 0$, whence it follows that

$$e^2k_\perp^2 I_{k_\perp} \simeq \frac{n^2}{z^{5/2}}\,T_i^2, \qquad \sqrt{z} \gg \frac{T_i}{T_e}\frac{n}{\varkappa\rho_i}. \tag{17}$$

Since the intensity of the established oscillations falls rather rapidly with increasing $k_\perp$, and for $k > k_0$ the oscillations are stable, the characteristic wave numbers are of order $k \simeq k_0$.

The intensity of the established cyclotron oscillations is above thermal, and they can be detected experimentally. (It is interesting to note that

ion cyclotron oscillations were observed in (5).) The interaction of plasma particles with cyclotron oscillations, generally speaking, should lead to anomalous diffusion. However, owing to the small scale of the oscillations, the diffusion rate is relatively small. The diffusion flux can be estimated with the aid of the kinetic equation for the averaged electron distribution function in the quasilinear approximation:

$$\frac{\partial f_0}{\partial t} + \mathbf{v}\nabla f_0 + [\mathbf{v},\omega_B]\,\frac{\partial f_0}{\partial \mathbf{v}} = -\frac{\pi e^2}{mT}\frac{\partial}{\partial \mathbf{v}}f_0\int \mathbf{k}\omega_k I_k \delta(\omega_k - k_z v_z)\,d\mathbf{k}. \tag{18}$$

Integrating this equation with weight $v_x$, we obtain a relation between the diffusion flux and the correlation function $I_k$

$$N\overline{v}_y = \int f_0 v_y\,d\mathbf{v} = \frac{\sqrt{\pi}ce^3N}{T_e eBv_{T_e}}\int \frac{k_x}{k_z}\,(k_x v_e^* - \omega_k)\,I_k e^{-(\omega_k/k_z v_{T_e})^2}\,d\mathbf{k} \simeq \frac{ce^2N}{eBT_e}\,k_\perp^3 I_{k\perp}. \tag{19}$$

Then the diffusion coefficient $D \simeq \overline{v}_y/\varkappa$ is, in order of magnitude, equal to

$$D \simeq \frac{v_{T_i}^2}{\omega_{B_i}}\left(\frac{v_e^*}{v_{T_i}}\right)^3 \equiv D_B\frac{T_i}{T_e}\left(\frac{v_e^*}{v_{T_i}}\right)^3. \tag{20}$$

(The quantity $D_B$ is customarily called the Bohm diffusion coefficient.) Thus, the drift-cyclotron instability leads to a diffusion coefficient, small in comparison with $D_B$, in the ratio $(\rho_i\varkappa)^3$. Comparison of (20) with the estimate of the diffusion coefficient due to the low-frequency drift instability $D_0$ (2, 3) gives

$$\frac{D}{D_0} \simeq (\varkappa\rho_i)^2\beta_e\frac{m_i}{m_e}, \tag{21}$$

which, by virtue of the restrictions used in the present work, does not exceed unity. (The opposite limiting case $(\varkappa\rho_i)^2\beta > m_e/m_i$ requires additional investigation.) In contrast to drift instability, which is possible only for a large ratio of the longitudinal and transverse dimensions of the experimental apparatus, $L/a \gg 10$, drift-cyclotron instability is possible also in short devices, provided only that $L/\rho_i \gg (m_i/m_e)^{1/2}$.

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## REFERENCES

1. A. B. Mikhailovskii, *Problems of Plasma Theory*, issue 3, 1963, p. 141.

2. B. B. Kadomtsev, ZhETF, **45**, 1230 (1963).

3. A. A. Galeev, L. I. Rudakov, ZhETF, **45**, 647 (1963).

4. A. B. Mikhailovskii, A. V. Timofeev, ZhETF, **44**, 919 (1963).

5. M. V. Babykin, P. P. Gavrin et al., ZhETF, **46**, 511 (1964).

*Note: Figure translations are in progress. See original paper for figures.*