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## Abstract

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## MATHEMATICS

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# A THEOREM ON THE DENSITY OF FINITE FUNCTIONS IN THE SPACE $L_{p,a}^{(m)}(E_n)$

*(Presented by Academician S. L. Sobolev on 10 IX 1963)*

Let $E_n$ be the $n$-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. Consider the set of functions $f(x)$ for which the relation

$$\|f\|^p_{L^{(m)}_{p,a}(E_n)} = \int_{E_n} a(x) \left[ \sum_{|\alpha|=m} |D^\alpha f|^2 \right]^{p/2} dx < \infty, \tag{1}$$

holds, where $a(x)$ is a measurable, almost everywhere finite and almost everywhere positive function, called the weight; $D^\alpha f = \partial^m f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ is the generalized partial derivative of order $m > 0$.

It is clear that (1) is equivalent to the following condition:

$$\int_\sigma d\sigma \left[ \int_0^\infty b(x) \left[ \sum_{|\alpha|=m} |D^\alpha f|^2 \right]^{p/2} dr \right] < \infty, \tag{2}$$

where $\sigma$ is the unit sphere of the space $E_n$,

$$b(x) = r^{n-1} a(x) = \left[ \sum_{i=1}^n x_i^2 \right]^{(n-1)/2} a(x). \tag{3}$$

Introduce the notation:

$$B(x) = b^{1/(1-p)}(x) = a^{1/(1-p)}(x)\, r^{(n-1)/(1-p)}. \tag{4}$$

For any $m > 0$ we shall subject the weight $a(x)$ to the following two conditions:

1.

$$\operatorname*{vrai\,inf}_{|x| \le R} a(x) = \varepsilon(R) > 0 \quad \text{for all} \quad 0 < R < \infty, \tag{5}$$

$$\operatorname*{vrai\,sup}_{|x| \le R} a(x) = \varepsilon_1(R) > 0 \quad \text{for all} \quad 0 < R < \infty.$$

2. There exists an $R_0 > 0$ such that, for $r \geq R_0$, $a(x)$ depends only on $r$, i.e.

$$a(x) = a(r). \tag{6}$$

In addition to conditions (5) and (6), we shall assume that: for $m = 1$, $a(r)$ is infinitely differentiable if $r \geq R_0$; for $m > 1$, $a(r)$ satisfies the conditions

$$\int_{R_0}^{r} B(t)\,dt < \infty \quad (R_0 < r < \infty); \tag{7}$$

$$\lim_{r\to\infty} \frac{B(r)}{\displaystyle\int_{R_0}^{r} B(t)\,dt} = \delta_1 > 0, \quad \text{if } \int_{R_0}^{\infty} B(t)\,dt = \infty; \tag{8}$$

$$\lim_{r\to\infty} \frac{B(r)}{\displaystyle\int_{r}^{\infty} B(t)\,dt} = \delta_2 > 0, \quad \text{if } \int_{R_0}^{\infty} B(t)\,dt < \infty. \tag{9}$$

The set $L_{p,a}^{(m)}(E_n)$ is a normed space of classes of functions. The zero class consists of polynomials of degree $m - 1$. Classes

are composed in the usual way. We shall call a class **finitely infinitely differentiable** if it contains a function possessing the indicated property.

Our task, taking into account the restrictions imposed on the weight above, is to prove the following theorem.

**Theorem.** *In the space $L_{p,a}^{(m)}(E_n)$, finitely infinitely differentiable classes form a dense set.*

We first prove one lemma.

**Lemma.** *Let $X$ and $Y$ be spaces with $\sigma$-finite measure, and let $b(x)$, $m(x, y)$, $K(x, y)$ be measurable functions such that the following relations hold almost everywhere:*

$$0 < b(x) < \infty; \tag{10}$$

$$0 < m(x, y) < \infty; \tag{11}$$

$$0 \leq K(x, y) < \infty; \tag{12}$$

$$0 < \int_Y K(x,y)m(x,y)\,dy < \infty; \tag{13}$$

$$0 < \int_X b^{1/(1-p)} K^{p'/q'} m^{-p'/q} \left[\int_Y Km\,dy\right]^{p'/q} dx < \infty. \tag{14}$$

*Then, for any nonnegative function $f(x)$ measurable on $X$, the inequality*

$$\left\{ \int_Y \left[ \int_X b^{1/(1-p)}(x) K^{p'/q'}(x,y) m^{-p'/q}(x,y) \left[\int_Y K(x,y)m(x,y)\,dy\right]^{p'/q} dx \right]^{-q/p'} \right.$$

$$\left. \times \left[\int_X K(x,y)f(x)\,dx\right]^{qdy} \right\}^{q/p} \le \left[\int_X bf^{pdx}\right]^{q/p}, \tag{15}$$

*where*

$$1 < p \le q < \infty. \tag{16}$$

If we now prove the density of finite classes in the space $L_{p,a}^{(m)}(E_n)$, the assertion of the theorem will follow from this, since every function $f(x) \in L_{p,a}(E_n)$ finite on $(E_n)$ can be obtained as the limit, in the metric of $L_{p,a}(E_n)$, of a sequence of infinitely differentiable finite functions, analogously to how this is done in ($^1$), p. 20.

We now pass directly to the proof of the theorem.

First consider the case $m = 1$. Obviously, only two possibilities can arise:

$$1. \qquad \int_{R_0}^{\infty} b^{1/(1-p)}(r)\,dr = \int_{R_0}^{\infty} B(r)\,dr < \infty. \tag{17}$$

$$2. \qquad \int_{R_0}^{\infty} B(r)\,dr = \infty. \tag{18}$$

If (17) holds, then it can be shown that $f(x)$ is representable in the form

$$f(r,\gamma) = C - \int_r^{\infty} \frac{\partial f(r,\gamma)}{\partial r}\,dr, \tag{19}$$

where $r = |x|$, $\gamma = x/|x|$, and $C$ does not depend on $\gamma$. Setting $C = 0$, we obtain the function

$$f(r,\gamma) = -\int_r^\infty \frac{\partial f(r,\gamma)}{\partial r}\,dr, \tag{20}$$

equivalent to the original one.

For the proof of the theorem it is sufficient to approximate the function (20) by finite functions in the metric of the space $L^{(1)}_{p,a}(E_n)$.

We introduce the cutoff function

$$\psi_\eta(r) = \psi(\eta\cdot\mu(r)), \tag{21}$$

where $\psi(\alpha)$ is an arbitrarily smooth function satisfying the condition

$$\psi(\alpha) = \begin{cases} 1, & \text{if } \alpha < 1/2, \\ 0, & \text{if } \alpha > 1; \end{cases} \tag{22}$$

$$\mu(r) = \left[\int_r^\infty B(t)\,dt\right]^{-1} \qquad (r > R_0). \tag{23}$$

We shall show that for the function (20) the equality

$$\lim_{\eta\to 0} \|f - f\cdot\psi_\eta(r)\|_{L^{(1)}_{p,a}(E_n)} = 0 \tag{24}$$

holds. We have:

$$\|f - f\cdot\psi_\eta(r)\|_{L^{(1)}_{p,a}(E_n)} \le K \sum_{|\alpha|=1} \|D^\alpha f - \psi_\eta(r)\cdot D^\alpha f - f D^\alpha \psi_\eta(r)\|_{L_{p,a}(E_n)} \le$$

$$\le K\left[\sum_{|\alpha|=1} \|(1-\psi_\eta(r))D^\alpha f\|_{L_{p,a}(E_n)} + \sum_{|\alpha|=1} \|f D^\alpha \psi_\eta(r)\|_{L_{p,a}(E_n)}\right]. \tag{25}$$

Denote by $\mu^{-1}(r)$ the function inverse to $\mu(r)$ $(r > R_0)$. We shall have:

$$\lim_{\eta\to 0} \|(1-\psi_\eta(r))D^\alpha f\|_{L_{p,a}(E_n)} = \lim_{\eta\to 0} \int_{|x|>\mu^{-1}(1/2\eta)} a(x)|D^\alpha f|^p\,dx = 0. \tag{26}$$

Introduce the function

$$\gamma(r) = \left[\frac{\mu'(r)}{\mu(r)}\right]^p . \tag{27}$$

Then

$$\|fD^\alpha\psi_\eta(r)\|_{L_{p,a}(E_n)} = \int_{|x|>R_0} a(r)|f(x)|^p|D^\alpha\psi_\eta(r)|^p\,dx \leq$$

$$\leq K_0\int_\sigma d\sigma\left[\int_{\mu^{-1}(1/\eta)}^\infty b(r)\gamma(r)\left(\int_r^\infty\left|\frac{\partial f}{\partial r}\right|dr\right)^p dr\right]. \tag{28}$$

The right-hand side of inequality (28) tends to 0 as $\eta\to 0$. Indeed, putting in (15) $X = Y = (R_0,\infty)$, $p = q$,

$$K(x,y) = \begin{cases} 1, & \text{if } x\geq y,\\ 0, & \text{if } x<y; \end{cases}$$

$$m(x,y) = \frac{d}{dy}\left[\left(\int_y^\infty B(t)\,dt\right)^{-1/p} - \left(\int_{R_0}^\infty B(t)\,dt\right)^{-1/p}\right]^{p-1}, \tag{29}$$

we obtain

$$\int_{R_0}^\infty b(r)\gamma(r)\left[\int_r^\infty\left|\frac{\partial f}{\partial r}\right|dr\right]^p dr \leq N\int_{R_0}^\infty b(r)\left|\frac{\partial f}{\partial r}\right|^p dr. \tag{30}$$

Integrating both sides of inequality (30) over the unit sphere and taking into account that $f\in L^{(1)}_{p,a}(E_n)$, we obtain

$$\int_\sigma d\sigma\left[\int_{R_0}^\infty b(r)\gamma(r)\left|\int_r^\infty\frac{\partial f}{\partial r}\,dr\right|^p dr\right] < \infty. \tag{31}$$

From inequality (31) it follows that the right-hand side of inequality (28) tends to 0.

Thus, the assertion of the theorem for the case $m = 1$ and

$$\int_{R_0}^\infty B(t)\,dt < \infty$$

has been proved. The remaining cases are proved on the basis of inequality (15) in an analogous way.

In the case $m = 1$ and when relation (18) is satisfied, we set

$$\psi_\eta(r) = \psi\left[\eta \cdot \int_{R_0}^{r} B(t)\,dt\right]. \tag{32}$$

In the case $m > 1$, we set

$$\psi_\eta(r) = \psi(\eta \cdot e^r). \tag{33}$$

The density theorem for finite functions in the space $L_{p,a}^{(m)}(E_n)$ for $a(x) \equiv 1$ was proved by S. L. Sobolev.

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*Note: Figure translations are in progress. See original paper for figures.*

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