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Abstract

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ON THE PROBLEM OF BRANCHING OF ALMOST-PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. Consider the system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n; \mu) \qquad (i = 1, 2, \dots, n), \tag{1}$$

which we shall write in vector form as

$$\frac{dx}{dt} = f(t, x; \mu). \tag{2}$$

Here μ is a scalar parameter. We shall assume that the right-hand sides of the system are almost-periodic in t and sufficiently smooth in x, μ.

Let $x_0(t) = \{x_1^0(t), \dots, x_n^0(t)\}$ be an almost-periodic solution of system (1) for $\mu = \mu_0$. We shall be interested in the question of the existence and number of almost-periodic solutions $x(t)$ of system (1), close to $x_0(t)$, for values of μ close to μ_0. As is known, this question is essentially connected with the properties of the solutions of the variational system

$$\frac{dy}{dt} = A(t)y, \tag{3}$$

where $A(t)$ is the matrix with elements

$$a_{ij}(t) = \frac{\partial f_i(t, x_1^0(t), \dots, x_n^0(t); \mu_0)}{\partial x_j}.$$

If zero is not a point of the spectrum of the problem on almost-periodic solutions of system (3), then, generally speaking, system (1), for values of μ close to μ_0, has a unique family of solutions $x(t, \mu)$, continuously depending on μ, which for $\mu = \mu_0$ turns into $x_0(t)$. This question, under various conditions, has been studied in detail by a number of authors (see, for example, ($^{1-4}$)).

We are interested in the case when the variational equations are degenerate in the sense that they have almost-periodic solutions. By analogy with problems on periodic solutions, one may expect that in this case, as μ varies, the solution $x_0(t)$ will branch into several distinct almost-periodic solutions. The question of the number of such solutions is naturally connected with branching equations analogous to the corresponding equations in the theory of nonlinear integral equations (see, for example, ($^{5-8}$)).

In the further constructions we restrict ourselves to the case when equations (3) are a system with constant coefficients, i.e.,

$$\frac{dx}{dt} = Ax, \tag{4}$$

where A is a constant matrix. It is assumed that the matrix A has eigenvalues on the imaginary axis.

2. Denote by A_1 the matrix $A - aE$, where E is the identity matrix and a is a sufficiently large number. All eigenvalues of the matrix A_1 lie in the left half-plane.

Let us introduce into consideration the nonlinear integral operator (see, for example, [3, 9, 10])

$$F(x, \mu) = \int_{-\infty}^{t} e^{(t-s)A_1} \left[f(s, x(s); \mu) - A_1 x(s) \right] ds. \tag{5}$$

This operator transforms almost-periodic functions into almost-periodic functions. In other words, this operator acts in the space B of almost-periodic functions (in the space B we consider the uniform norm

$$\|x(t)\|_B = \sup_{-\infty<t<\infty} \sum_{i=1}^{n} |x_i(t)|.$$

A simple calculation shows that the fixed points of the operator $F(x, \mu)$ in the space B are the almost-periodic solutions of system (1). Thus, the question of branching of almost-periodic solutions of system (1) coincides with the question of branching of solutions in B of the nonlinear integral equation

$$x(t, \mu) = F(x, \mu). \tag{6}$$

Unfortunately, the operator $F(x,\mu)$ does not possess the property of complete continuity, which makes it difficult to apply general methods of nonlinear functional analysis to its study.

3. The operator $F(x,\mu)$ is Fréchet differentiable at each point of the space B. It is easy to see that the derivative $K = F_x(x_0,\mu_0)$ has the form

$$Kx(t) = \alpha \int_{-\infty}^{\infty} e^{(t-s)A_1} x(s)\, ds. \tag{7}$$

We shall assume that the matrix A has a zero eigenvalue. Then 1 is an eigenvalue of the operator (7). This eigenvalue is at the same time a point of the continuous spectrum of this operator. More precisely, the spectrum S_k of the operator (7), considered in B, coincides with the set of all solutions λ of all equations

$$\det \left| A + \left[\alpha \left(\frac{1}{\lambda} - 1 \right) + i\beta \right] E \right| = 0 \qquad (-\infty < \beta < \infty).$$

At the same time the operators $K - \lambda_0 I$, for $\lambda_0 \in S_k$, do not possess the property of normal solvability.

Let γ be a fixed positive number. Denote by B_γ the collection of those vector-functions $x(t) \in B$ whose components correspond to Fourier series

$$x_k(t) \sim \sum_{n=0}^{\infty} c_n^{(k)} e^{i\omega_n t} \qquad (k = 1, 2, \dots, n),$$

where $\omega_0 = 0,\ \omega_n \geq \gamma\ (n \geq 1)$.

The operator (7) leaves invariant each subspace B_γ $(\gamma > 0)$. On passing to the subspace B_γ, the spectrum of the operator (7) decreases.

Theorem 1. *Let γ be greater than all positive eigenvalues of the matrix $-iA$. Then 1 is an isolated eigenvalue of the operator K considered on B_γ. Moreover, the operator $K - I$ on B_γ is normally solvable and its index is equal to zero.*

Under the conditions of Theorem 1, the multiplicity of the eigenvalue 1 of the operator K in B_γ coincides with the multiplicity of the zero eigenvalue of the matrix A. In what follows, for simplicity, we shall assume that the eigenspace $E_0 \subset B_\gamma$ of the operator K, corresponding to the eigenvalue 1, consists only of eigenvectors. The subspace E_0 consists

of constant vector-functions; the values of these vector-functions are eigenvectors of the matrix A corresponding to the zero eigenvalue.

Let $e_1, \dots, e_r$ be a basis in the subspace of eigenvectors of the matrix A corresponding to the zero eigenvalue, and let $g_1, \dots, g_r$ be a biorthogonal $((e_i, g_j) =$

δ_{ij}) system of eigenvectors of the adjoint matrix, also corresponding to the zero eigenvalue. Define on B the linear operator P by the equality

$$Px = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{i=1}^{r} (x(s), g_i) e_i \, ds. \tag{8}$$

Theorem 2. *The operator (8) commutes with the operator (7). The set of values of the operator $K - I$ coincides with the set of values of the operator $Q = I - P$. The operator P projects onto the subspace E_0 along the range E_1 of the operator $K - I$.*

4. We return to the study of system (1).

Theorem 3. *Let the vector-function $f(t, x; \mu)$ satisfy the following essential restriction: $f(t, x; \mu) \in B_\gamma$ for fixed x and μ. In addition, we shall assume that the right-hand sides of system (1) are analytic in the spatial variables in the region in which the values of the solution $x_0(t)$ lie. Then the operators (5) are defined in some neighborhood of the point x_0 in B_γ, and their values belong to B_γ.*

Equation (6) can be written as the system of two nonlinear equations

$$u + Px_0 - PF(u + v + x_0, \mu) = 0, \tag{9}$$

$$(K - I)(v + Qx_0) = K(v + Qx_0) - QF(u + v + x_0, \mu), \tag{10}$$

where $u = Px - Px_0,\ v = Qx - Qx_0$.

Theorem 4. *Let the assumptions of Theorem 3 be satisfied. Then equation (10), for values of u and μ close to u_0 and for small u, has a unique small solution $v = R(u, \mu)$ in B_γ, which vanishes for $\mu = \mu_0$.*

It follows from Theorem 4 that the question of the existence of almost-periodic solutions of system (1) (belonging to B_γ) near $x_0(t)$ is equivalent to the question of the existence of small solutions u of equation (9). The solution u of this equation may be sought in the form $u = \sum_{i=1}^{r} \xi_i e_i$. To determine the coefficients $\xi_1, \dots, \xi_r$, we obtain the system of scalar equations

$$\sum_{i=1}^{r} \xi_i e_i + Px_0 - PF\left(\sum_{i=1}^{r} \xi_i e_i + R\left(\sum_{i=1}^{r} \xi_i e_i, \mu\right) + x_0, \mu\right) = 0. \tag{11}$$

System (11) is precisely the branching equation. Consequently, the question of the number of solutions near $x_0(t)$ is reduced to the analysis of system (11).

5. If the matrix A has no zero eigenvalues, then the change of variables

$$x(t) = e^{i\nu Et} y(t),$$

where ν is a positive eigenvalue of the matrix $-iA$, reduces this case to the one already studied.

As an example, consider the differential equation

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) + a(t)x^2(t) + b(t)x^3(t) + \mu[c(t) + d(t)x(t)] + \mu^2 e(t) = 0, \quad (12)$$

which, for $\mu = 0$, has the solution $x(t) \equiv 0$. Branching of periodic solutions of equation (12) was studied in the work (^{8}). We shall consider the question of almost-periodic solutions.

Theorem 5. Let γ be an arbitrary positive number. Let $a(t)e^{i\omega t} \in B_\gamma,\ b(t)e^{2i\omega t} \in B_\gamma,\ c(t)e^{-i\omega t} \in B_\gamma,\ d(t) \in B_\gamma,\ e(t)e^{-i\omega t} \in B_\gamma$. Then, if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\omega s} c(s)\, ds \neq 0; \qquad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a(s) e^{i\omega s}\, ds \neq 0,$$

then equation (12) has, for small nonzero μ, two small almost-periodic solutions belonging to $e^{i\omega t} B_\gamma$.

If

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c(s) e^{-i\omega s}\, ds \neq 0, \qquad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a(s) e^{i\omega s}\, ds = 0,$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} b(s) e^{2i\omega s}\, ds \neq 0,$$

then equation (12) has, for small nonzero μ, three small almost-periodic solutions belonging to $e^{i\omega t} B_\gamma$.

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Note: Figure translations are in progress. See original paper for figures.

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