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Abstract

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MATHEMATICS

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ON THE FOURIER TRANSFORM OF A SLOWLY DECREASING FUNCTION

(Presented by Academician A. N. Kolmogorov, 3 VIII 1963)

1. In the works of A. N. Tikhonov and A. A. Samarskii⁽¹⁻³⁾, a method was proposed for regularizing divergent integrals arising in the study of the asymptotics of integrals of the form:

$$I(h) = \int_0^{\infty} f(hx) \omega(x) dx \quad (h \rightarrow 0), \quad (1)$$

where the kernel $\omega(x)$ has, as $x \rightarrow \infty$, the asymptotics

$$\omega(x) = \frac{q_1}{x} + \frac{q_2}{x^2} + \dots + \frac{q_n}{x^n} + o\left(\frac{1}{x^n}\right),$$

q_1, q_2, \dots, q_n are numbers.

Applying this method, one can prove the following theorem:

Theorem 1. *If the function $\omega(x)$, as $x \rightarrow \infty$, has the asymptotics*

$$\omega(x) = \begin{cases} \frac{q_1^+}{x} + \frac{q_2^+}{x^2} + \dots + \frac{q_N^+}{x^N} + \omega_N^+(x), & x \rightarrow +\infty, \\ \frac{q_1^-}{x} + \frac{q_2^-}{x^2} + \dots + \frac{q_N^-}{x^N} + \omega_N^-(x), & x \rightarrow -\infty, \end{cases} \quad (2)$$

where

$$|\omega_N^{\pm}(x)x^N| \xrightarrow{x \rightarrow \infty} 0, \quad \int_{\mp\infty}^{\mp A} |\omega_N(x)x^{N-1}| dx < \infty,$$

then its Fourier transform

$$\tilde{\omega}(t) = \int_{-\infty}^{\infty} e^{itx} \omega(x) dx$$

as $t \rightarrow 0$ has an asymptotics of the form

$$\tilde{\omega}(t) = \sum_{k=0}^{N-1} a_k(t) \frac{(it)^k}{k!} + o(t^{N-1}), \quad t > 0,$$

where

$$a_k(t) = -(q_{k+1}^+ - q_{k+1}^-) \ln t + \Omega_k + (q_{k+1}^+ - q_{k+1}^-) \left(\sum_{\nu=1}^k \frac{1}{\nu} - c \right) + \frac{i\pi}{2} (q_{k+1}^+ + q_{k+1}^-),$$

$$\Omega_k = \int_{-\infty}^{-1} \left[\bar{\omega}_k^-(x) - \frac{q_{k+1}^-}{x} \right] dx + \int_{-1}^0 \bar{\omega}_k^-(x) dx + \int_0^1 \bar{\omega}_k^+(x) dx + \int_1^{\infty} \left[\bar{\omega}_k^+(x) - \frac{q_{k+1}^+}{x} \right] dx,$$

$$\bar{\omega}_{k+1}^{\pm}(x) = x \bar{\omega}_k^{\pm}(x) - q_{k+1}^{\pm}, \quad \bar{\omega}_0^{\pm}(x) = \omega(x), \quad c = 0.5772 \dots$$

Developed methods make it possible to compute the asymptotics of $\tilde{\omega}(t)$ as $t \rightarrow 0$ also when the expansion (2) contains fractional powers of x ; however, in this case more cumbersome formulas are obtained, and we shall not write them out here.

2. The theorem proved has applications in probability theory. With its help one can generalize Cramér's well-known expansion of the distribution function of the sum of independent random variables to the case when the summands have no moments.

Let $\xi_1, \xi_2, \dots, \xi_n$ be independent identically distributed continuous random variables with mathematical expectation 0 and variance 1. Let $\omega(x)$ be the probability density of the random variable ξ , and suppose it has the asymptotics (2). Consider the sum

$$\sigma_n = \frac{\xi_1 + \xi_2 + \dots + \xi_n}{\sqrt{n}}$$

and let $F_n(x)$ be the distribution function of the random variable σ_n .

Introduce the functions

$$\eta_j(t) = \frac{d}{dx^j} \ln \left[1 + \sum_{m=1}^{N-1} a_m(t) \frac{x^m}{m!} \right] \Bigg|_{x=0}, \quad t > 0;$$

$$\beta_j(t; \xi) = \frac{d}{dx^j} \exp \left[-t^2 \sum_{m=1}^{N-1} \frac{\eta_m(\xi) x^{m-2}}{m!} \right] \Big|_{x=0};$$

$$g(t; \xi) = e^{-0.5t^2} \left[1 + \sum_{m=1}^{N-3} \beta_m(t; \xi) \frac{(i\xi)^m}{m!} \right];$$

$$G_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx}}{(-it)} g \left(t; -\frac{t}{\sqrt{n}} \right) dt \quad \left(g \left(-t; -\frac{t}{\sqrt{n}} \right) = g^* \left(|t|; \left| \frac{t}{\sqrt{n}} \right| \right) \right).$$

Theorem 2. *If the assumptions stated above are fulfilled, then the estimate*

$$|F_n(x) - G_n(x)| = O \left(\frac{1}{n^{(N-3)/2}} \right).$$

is valid.

3. The method of A. N. Tikhonov and A. A. Samarskii also extends to kernels having, as $x \rightarrow \infty$, asymptotics of the form

$$\omega(x) = \frac{e^{ix}}{\sqrt{x}} \left(a_1 + \frac{a_2}{x} + \dots + \frac{a_{N+1}}{x^N} \right) + O \left(\frac{1}{x^{N+1}} \right). \quad (3)$$

Let

$$f_{m+1}(x) = \frac{f_m(x) - f_m(0)}{x}, \quad f_0(x) = f(x),$$

$$R^0 f(x) = f(x), \quad R^1 f(x) = f'(x) - 0.5 \frac{f(x) - f(0)}{x}, \quad R^m f(x) = R^1 \{ R^{m-1} f(x) \},$$

$$D_m(\omega) = \int_0^{\infty} x^m \left[\omega(x) - \frac{e^{ix}}{\sqrt{x}} \left(a_1 + \frac{a_2}{x} + \dots + \frac{a_{m+1}}{x^m} \right) \right] dx + \\ + \sqrt{\pi i} \sum_{k+j=m} 0.5 \cdot 1.5 \dots (k-0.5) (i)^k a_{j+1}.$$

Theorem 3. *If:*

- 1) $\omega(x)$ has the asymptotic form (3);
- 2) $f(x)$ has N right derivatives at the point $x = 0$;

3)

$$\int_0^\infty \frac{1}{\sqrt{x}} |R^{(N-m)} f_m(x)| dx < \infty,$$

then for the integral (1) the asymptotic formula

$$I(h) = \sum_{m=0}^N D_m(\omega) \frac{f^{(m)}(0)}{m!} h^m + o(h^N)$$

holds.

Using Theorem 3, one can obtain the asymptotic expansion of integrals containing the Bessel function of order ν .

Theorem 4. If the hypotheses of Theorem 3 are satisfied for $f(x)$, then the integral

$$F_\nu(\lambda) = \int_0^\infty f(x) J_\nu(\lambda x) dx$$

as $\lambda \rightarrow \infty$ has the asymptotic form

$$F_\nu(\lambda) = \sum_{m=0}^N Q_m(\nu) f^{(m)}(0) \left(\frac{1}{\lambda}\right)^{m+1} + o\left(\frac{1}{\lambda^{N+1}}\right),$$

where

$$Q_m(\nu) = \frac{(-1)^m}{m!} \frac{d}{dx^m} \left[\frac{1}{(1+x^2)^{0.5} (x + \sqrt{1+x^2})^\nu} \right]_{x=0}.$$

The asymptotic formula for $F_0(\lambda)$ was obtained in (4) by a somewhat different method. Putting $f(x) = x\tilde{f}(x)$ in Theorem 4, we obtain the asymptotic formula for the Hankel transform of the function $\tilde{f}(x)$.

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Note: Figure translations are in progress. See original paper for figures.

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