ON AN ANALOG OF THE DUHAMEL INTEGRAL FOR PRE-EXCITED LINEAR FILTERS WITH VARIABLE PARAMETERS

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Abstract

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ON AN ANALOG OF THE DUHAMEL INTEGRAL FOR PRE-EXCITED LINEAR FILTERS WITH VARIABLE PARAMETERS

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Let us have a linear filter (electrical system) with variable parameters, i.e., one whose output voltage $y(t)$ is related to the input voltage $x(t)$ by an ordinary differential equation of order n:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1(t)\frac{dy}{dt} + a_0(t)y = x(t), \qquad (1)$$

where $a_i(t)$ $(i = 1, 2, \dots, n)$ are variable continuous parameters $(a_n(t) \neq 0)$, and $x(t)$ and $y(t)$ are regarded as continuous deterministic (nonrandom) quantities. If the filter is initially unexcited, i.e., at time $t = 0$ we have the initial conditions

$$\left.\frac{d^\nu y}{dt^\nu}\right|_{t=0} = 0, \quad \nu = 0, 1, 2, \dots, (n-1), \qquad (2)$$

then, as is known ([1]), the output voltage $y(t)$ is related to the input voltage $x(t)$ by the relation

$$y(t) = \int_0^t w(t-u, u)x(u)\,du, \quad t \geqslant 0. \qquad (3)$$

We shall call (3) the Duhamel integral for initially unexcited linear filters with variable parameters. Here $w(t-u, u)$ is the "weight" function (the response of the filter to an input signal in the form of a delta function). In the theory of differential equations, $w(t-u, u)$ is called the Cauchy function ([2]).

However, the filter is often an initially excited system, i.e., one with nonzero initial conditions:

$$\left.\frac{d^\nu y}{dt^\nu}\right|_{t=0} = y_0^{(\nu)}, \quad \nu = 0, 1, 2, \dots, (n-1). \qquad (4)$$

Let us determine in this case the relation between $y(t)$ and $x(t)$. Introduce a new function $z(t)$, equal to

$$z(t) = y(t) - \sum_{\nu=0}^{n-1} \frac{y_0^{(\nu)}}{\nu!}\, t^{\nu}. \tag{5}$$

Then equation (1) is reduced to the form

$$a_n(t)\frac{d^n z}{dt^n} + a_{n-1}(t)\frac{d^{n-1} z}{dt^{n-1}} + \dots + a_1(t)\frac{dz}{dt} + a_0(t)z = F(t), \tag{6}$$

where $F(t)$ denotes the function

$$F(t) = x(t) - a_0(t)\sum_{\nu=0}^{n-1} \frac{y_0^{(\nu)}}{\nu!}\, t^{\nu} - a_1(t)\sum_{\nu=1}^{n-1} \frac{y_0^{(\nu)}}{(\nu-1)!}\, t^{\nu-1} - \cdots$$

$$\cdots - a_{n-1}(t) y_0^{(n-1)}. \tag{7}$$

For equation (6) we have zero initial conditions

$$\left.\frac{d^{\nu} z}{dt^{\nu}}\right|_{t=0} = 0, \qquad \nu = 0, 1, 2, \dots, (n-1). \tag{8}$$

Then the function $z(t)$ is related to the function $F(t)$, which is the transformed voltage at the input of the initially unexcited filter (6), by a "convolution" integral with the "weight" function $h(t-u, u)$:

$$z(t) = \int_0^t h(t-u, u) F(u)\, du, \qquad t \geqslant 0. \tag{9}$$

Taking (5) into account, from relation (9) we obtain the expression

$$y(t) = \sum_{\nu=0}^{n-1} \frac{y^{(\nu)}}{\nu!} t^{\nu} + \int_0^t h(t-u, u) F(u)\, du, \qquad t \geqslant 0. \tag{10}$$

If, however, the filter is initially unexcited, then the initial conditions (2) are realized, and from (10) it follows that

$$y(t) = \int_0^t h(t-u, u) x(u)\, du, \qquad t \geqslant 0. \tag{11}$$

From (3) and (11) we obtain

$$w(t-u,u) \equiv h(t-u,u), \tag{12}$$

i.e., finally in (10) we have

$$y(t) = \sum_{\nu=0}^{n-1} \frac{y^{(\nu)}}{\nu!} t^{\nu} + \int_0^t w(t-u,u) F(u)\, du, \qquad t \geqslant 0. \tag{13}$$

We shall call this expression Duhamel' s integral for an initially excited filter. When using expression (13), as in the case of a filter with constant parameters, the following practical problem may often arise: it is required to find the "weight" function $w(t-u,u)$ from the known output and input voltages $y(t)$ and $x(t)$ of an initially excited system. We shall briefly describe a method for solving this problem.

Using the function $z(t)$, it is not difficult to reduce this problem to finding the kernel $h(t-u,u) \equiv w(t-u,u)$ of equation (9). On the basis of the relation between an ordinary linear differential equation and a linear Volterra integral equation of the second kind ([3,4]), we conclude that the transformed voltage $z(t)$, under the initial conditions (8), satisfying the differential equation (6) in a finite interval $[0,T]$ of variation of the time t, can be represented in the form

$$z(t) = \frac{1}{(n-1)!} \int_0^t (t-z)^{n-1} \varphi(z)\, dz, \qquad t \geqslant 0, \tag{14}$$

where $\varphi(z)$ is the solution of the Volterra integral equation of the second kind

$$\varphi(t) + \int_0^t K(t,z) \varphi(z)\, dz = f(t), \tag{15}$$

where the kernel $K(t,z)$ and the right-hand side $f(t)$ are respectively equal to

$$K(t,z) \equiv K(t-z,t) = \sum_{i=1}^{n} \frac{a_{n-i}(t)}{a_n(t)} \frac{(t-z)^{i-1}}{(i-1)!}, \tag{16}$$

$$f(t) = \frac{x(t)}{a_n(t)} - y_0^{(n-1)} \frac{a_{n-1}(t)}{a_n(t)} - (y_0^{(n-1)} t + y_0^{(n-2)}) \frac{a_{n-2}(t)}{a_n(t)} - \cdots$$

$$\cdots - \left(y_0^{(n-1)} \frac{t^{n-1}}{(n-1)!} + \cdots + y_0' t + y_0 \right) \frac{a_0(t)}{a_n(t)}. \tag{17}$$

From (9) and (14) it follows that the sought "weight" function $w(t-u,u)$ will be

$$w(t-u,u) = \frac{(t-u)^{n-1}\varphi(u)}{(n-1)!\,F(u)}, \qquad t \geq 0. \tag{18}$$

Consequently, in order to use formula (18), it is necessary first to solve the Volterra integral equation of the second kind (15) for the function $\varphi(t)$. In a number of cases it is possible to find an exact solution, but in most cases one finds an approximate solution $\widetilde{\varphi}(t)$ by one or another known approximate method. Then, correspondingly, we obtain an approximate value of the sought function

$$\widetilde{w}(t-u,u) = \frac{(t-u)^{n-1}\widetilde{\varphi}(u)}{(n-1)!\,F(u)}, \qquad t \geq 0, \tag{19}$$

which is an important characteristic of the voltage dynamics of the filter.

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Note: Figure translations are in progress. See original paper for figures.