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Abstract

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GENERAL BOUNDARY-VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL ELLIPTIC SYSTEMS

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In this note a class of singular integro-differential (s.i.d.) elliptic systems in a bounded domain G of n-dimensional Euclidean space R^n is described. To these systems there extends the general theory of elliptic problems with s.i.d. boundary conditions, constructed for differential elliptic systems in ($^{1-3}$). Deformations in the new, broader class of elliptic problems make it possible to reduce the computation of the index of an elliptic problem to the computation of the indices of two systems of singular integral (s.i.) equations in the case when, for the given elliptic system, the Dirichlet problem is Noetherian (see Theorems 2-5). In particular, simple conditions are obtained that are sufficient for the index of the Dirichlet problem to be equal to zero (see the corollary to Theorem 5). These results were mentioned in (^{4}).

1. The notation will be essentially the same as in (^{2}). The boundary Γ of the domain G is assumed to be an infinitely (or sufficiently) smooth $(n-1)$-dimensional surface admitting local straightening.

For an integer $k \geqslant 0$, denote by $H_k(G) = W_2^{(k)}(G)$ the space of column vectors $u(x)$ of height p, whose components have square-summable generalized derivatives of order $\leqslant k$ in G (^{5}). $H_k(R^n)$ is defined analogously. We shall denote the norms in these spaces by $\| \ \|_{k,G}$ and $\| \ \|_{k,R^n}$. Let $(s, m_1, \dots, m_r)$ be a certain collection of integers, $s > 0$, $m_\nu \geqslant 0$ (these will be the orders of the elliptic system and of the boundary operators). Fix an integer $l_1 \geqslant l_0 = \max(s, m_\nu + 1)$. Consider in the domain G the operator

$$Au(x) = M \sum_{|\alpha| \leqslant s} a_\alpha D^\alpha L u(x). \tag{1}$$

Here $x = (x^1, \dots, x^n)$; L is an operator of extension of functions from G to R^n, bounded in the norms $\| \ \|_k$, $0 \leqslant k \leqslant l_1$ (^{7}); M is the operator of restriction of functions from R^n to G; $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_\nu = -i\partial/\partial x^\nu$; $|\alpha| = \sum \alpha_\nu$. By a_α are denoted matrix s.i. operators (^{6}) on R^n (square matrices of order p). Their symbols $\sigma_\alpha(x,\xi)$ are assumed to be positively homogeneous of degree 0 in $\xi = (\xi_1, \dots, \xi_n)$ and to belong on the unit sphere Σ ($|\xi| = 1$) to the space $C_{qH_{n/2}}(\Sigma)$, where $q = l_1 - s$. It consists of functions (more precisely, of vector functions with p^2 components) which: 1) for each x belong to the space $H_{n/2}(\Sigma) = W_2^{(n/2)}(\Sigma)$ (^{8}); 2) have in it derivatives $D_x^\beta \sigma_\alpha(x,\xi)$ in the strong sense (i.e. in the norm $\| \ \|_{n/2,\Sigma}$ in $H_{n/2}(\Sigma)$), $|\beta| \leqslant q$, strongly continuous with respect to x; 3) satisfy the condition $\|\sigma_\alpha\|_{q,n/2} = \max_{x,\beta} \|D_x^\beta \sigma_\alpha\|_{n/2,\Sigma} < \infty$; 4) for large $|x|$ do not depend on x. The space $C_{qH_{n/2}}(\Sigma)$ is a normed ring. From the embedding theorem (^{8}) it follows that $D_x^\beta \sigma_\alpha(x,\xi)$ ($|\beta| \leqslant q$) are continuous in (x,ξ) for $\xi \neq 0$. Under the assumptions just stated, A is a bounded operator from $H_l(G)$ to $H_{l-s}(G)$ ($s \leqslant l \leqslant l_1$).

The **symbol** of the operator A will mean the matrix

$$\sigma_A(x,\xi) = \sum_{|\alpha|=s} \sigma_\alpha(x,\xi)\xi^\alpha$$

$$(x \in \overline{G},\ \xi \neq 0),$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. The operator A will be called **elliptic** if

$$\det \sigma_A(x,\xi) \neq 0 \quad (x \in \overline{G},\ \xi \neq 0).$$

For each point $X \in \Gamma$, the system of coordinates **associated** with X will mean the system obtained from the original system of coordinates x in R^n by translating the origin to X and rotating the axes so that the axis x^n acquires the direction of the inward normal to Γ at X.

We shall call the symbol σ_A **admissible** if, for each point $X \in \Gamma$, in the coordinate system associated with X, the matrix $\sigma_A(0,\xi)$ is a polynomial in ξ_n of degree s.

An elliptic operator A with admissible symbol will be called **properly elliptic** if, for each point $X \in \Gamma$, in the coordinate system associated with X, the ξ_n-roots of the polynomial $\det \sigma_A(0,\xi',\xi_n)$ for $\xi' \neq 0$ are distributed equally between the upper and lower half-planes. In this case the number ps is even: $ps = 2r$.

The operator A may be, in particular, any differential operator with smooth coefficients. (In this case $a_\alpha = \sigma_\alpha$ are simply matrices of smooth functions of x.) Then the operators L and M become unnecessary and may be omitted. The symbol of a differential operator is its characteristic matrix. The definitions of ellipticity and proper ellipticity then coincide with the generally accepted definitions, and proper ellipticity follows from ellipticity if $n > 2$ (^{9}).

The operator A depends on the choice of the operator L (which, for example, may contain multiplication by any finite sufficiently smooth function equal to 1 in a neighborhood of the domain G) and on lower-order terms. However, the following is true.

Lemma 1. *If A_1 and A_2 are two operators of the form (1) with one and the same symbol, and this symbol is admissible, then the difference $A_1 - A_2$ is a completely continuous operator from $H_l(G)$ to $H_{l-s}(G)$ $(s \le l \le l_1)$.*

On Γ let us prescribe boundary operators

$$B_\nu u = \sum_{|\beta| \le m_\nu} b_{\nu\beta} D^\beta u(x)\big|_\Gamma \quad (\nu = 1, \dots, r). \tag{2}$$

Here $b_{\nu\beta}$ are singular integral operators on Γ (rows of length p). The assumptions on their symbols are analogous to the assumptions concerning σ_α (with the natural changes) and are not given here for lack of space. The operators B_ν act boundedly from $H_l(G)$ into the spaces $H_{l-m_\nu-1/2}(\Gamma)^{(8)}$, $(m_\nu + 1 \le l \le l_1)$. The B_ν may be, in particular, differential operators with smooth coefficients; in this case $b_{\nu\beta} = \sigma_{\nu\beta}$ are rows of smooth functions on Γ.

The Ya. B. Lopatinskii condition (L) for a properly elliptic singular integro-differential operator A and for B is formulated in the same way as in the case of a differential A (2). For any point $X \in \Gamma$, in the coordinate system associated with X, put

$$\sigma_\nu(0, \xi) = \sum_{|\beta| = m_\nu} \sigma_{\nu\beta}(0, \xi')\xi^\beta$$

and consider the problem on the half-line

$$\sigma_A(0, \xi', D_n) v(x^n) = 0 \quad (x^n > 0); \tag{3}$$

$$\sigma_\nu(0, \xi', D_n) v(x^n) = h_\nu \quad (x^n = 0;\ \nu = 1, \dots, r). \tag{4}$$

Since σ_A is an admissible symbol, (3) is a system of ordinary differential equations. The space $\mathfrak{M}$ of its solutions tending to 0 as $x^n \to +\infty$ has dimension r by proper ellipticity. The condition (L) (at the point X) consists in the fact that, for $\xi' \ne 0$ and arbitrary numbers h_ν, the problem (3)–(4) must be uniquely solvable in $\mathfrak{M}$. This condition can be written in explicit form (see (9,2)).

Set $\mathfrak{A} = (A, B) = (A, B_1, \dots, B_r)$. We shall call the operator $\mathfrak{A}$ **elliptic** if A is properly elliptic and if A and B_ν are connected at each point $X \in \Gamma$ by condition (L).

The operator $\mathfrak{A}$ acts boundedly from $H_l(G)$ into the direct product $H_l(G, \Gamma)$ of the spaces $H_{l-s}(G)$ and $H_{l-m_\nu-1/2}(\Gamma)$ $(\nu = 1, \dots, r)$, for $l_0 \le l \le l_1$.

The following four theorems generalize the corresponding results of papers (2–3).

Theorem 1. *Let $\mathfrak{A}$ be an elliptic operator. Then: 1) for $u \in H_l(G)$ the a priori estimate holds*

$$\|u\|_{l,G} \le C \left(\|Au\|_{l-s,G} + \sum_{\nu=1}^{r} \|B_\nu u\|_{l-m_\nu-1/2,\Gamma} + \|u\|_{0,G} \right), \tag{5}$$

where C is a constant independent of u; 2) $\mathfrak{A}$ is a Φ-operator from $H_l(G)$ into $H_l(G,\Gamma)$ $(l_0 \le l \le l_1)$.

The latter means (^{10}) that the equation $\mathfrak{A}u = 0$ has a finite number α of linearly independent solutions in $H_l(G)$, that $\mathfrak{A}H_l(G)$ is closed in $H_l(G,\Gamma)$, and that the quotient space $H_l(G,\Gamma)/\mathfrak{A}H_l(G)$ has finite dimension β. The difference $\chi(\mathfrak{A}) = \alpha - \beta$ is called the **index** of the operator $\mathfrak{A}$. Under the assumptions of Theorem 1, the numbers α and β do not depend on l $(l_0 \le l \le l_1)$.

Let the operator $\mathfrak{A}_t$ depend on the parameter t, $0 \le t \le 1$. We shall say that $\mathfrak{A}_t$ depends **continuously** on t if the symbols σ_α are continuous with respect to t in the norm of $C_{qH_{n/2}}(\Sigma)$, and the symbols $\sigma_{\nu\beta}$ are continuous with respect to t in the norm in the corresponding spaces $(0 \le t \le 1)$.

Theorem 2. *Let $\mathfrak{A}_t$ be an elliptic operator depending continuously on t, $0 \le t \le 1$. Then the index $\chi(\mathfrak{A}_t)$ does not depend on t.*

Theorem 3. *Let (A, B') and (A, B'') be elliptic operators with the same A. Then*

$$\chi(A, B') - \chi(A, B'') = \chi(S),$$

where S is a certain system of r singular integral equations with r unknown functions on Γ.

The symbol of the system S is constructed in the same way as in (^{2}).

Theorem 4. *Let (A', B) and (A'', B) be elliptic operators with the same B, and suppose that the symbols $\sigma_{A'}$ and $\sigma_{A''}$ of the operators A' and A'' coincide for $x \in \Gamma$. Then*

$$\chi(A', B) - \chi(A'', B) = \chi(S),$$

where S is a system of p singular integral equations in R^n with p unknown functions. The symbol of the system S is equal to $\sigma_{A'} \cdot \sigma_{A''}^{-1}$ for $x \in G$ and to the identity matrix E for $x \notin G$.

2. Let A be a properly elliptic singular integro-differential operator of even order $s = 2m$. Take as B the operator corresponding to the Dirichlet problem:

$$u|_\Gamma = g_1, \dots, \left.\frac{\partial^{m-1}u}{\partial n^{m-1}}\right|_\Gamma = g_m, \tag{6}$$

where $\partial/\partial n$ is differentiation in the direction of the inner normal to Γ, and the g_j are columns of functions on Γ of height p.

Lemma 2. *Condition (L) for A, B at the point $X \in \Gamma$ is equivalent to the following condition: in the coordinate system associated with X,*

$$\sigma_A(0,\xi) = \sigma_-(\xi)\sigma_+(\xi), \tag{7}$$

where σ_- and σ_+ are matrices polynomial in ξ_n of degree m; for real $\xi' \ne 0$, the roots ξ_n of the polynomials $\det\sigma_+(\xi)$ and $\det\sigma_-(\xi)$ coincide with the roots *of the polynomial* $\det \sigma_A(0,\xi)$ *lying respectively in the upper and in the lower half-plane.*

This lemma, which in essence concerns ordinary differential equations, was proved in (^{11}); see also (^{12}).

Theorem 5. *Let* $\mathfrak{A}$ *be an elliptic operator corresponding to the Dirichlet problem (6). Then* $\chi(\mathfrak{A}) = \chi(S)$, *where* S *is a certain system of* p *singular integral equations in* R^n *with* p *unknown functions; its symbol is equal to* E *for* $x \notin G$.

For $x \in G$ this symbol is constructed as follows. Let $\sigma_\pm(\xi) = \sigma_\pm(\xi', \xi_n)$ in (7). Subject the symbol $\sigma_A = \sigma_0$ to a deformation on Γ: for each point $X \in \Gamma$, in the coordinate system associated with X, set

$$\sigma_t(0,\xi) = \sigma_-((1-t)\xi', \xi_n + it|\xi'|) \cdot \sigma_+((1-t)\xi', \xi_n - it|\xi'|).$$

$(0 \le t \le 1)$. We smoothly extend this deformation into the interior of the domain G. Let $\sigma_1(x,\xi)$ be the symbol thus obtained. Fix arbitrarily $\xi = \xi_0 \neq 0$ and set

$$\sigma(x,\xi) = \sigma_1^{-1}(x, \xi_0/|\xi_0|) \cdot \sigma_1(x, \xi/|\xi|).$$

This is the symbol of the operator S for $x \in G$.

The proof of Theorem 5 uses Theorems 2 and 4, as well as the fact that the Dirichlet problem for the metaharmonic operator $\Delta^m E$ has zero index (^{13}).

In (^{14}) a general formula was obtained expressing the index of a system of singular integral equations on a compact manifold without boundary in terms of its symbol. Therefore the index of the system S in Theorem 3 may be regarded as known. Using stereographic projection of the space R^n onto the n-dimensional sphere, one can compute the indices of the system S in Theorems 4 and 5. In particular, one obtains

Corollary. *Let* $\mathfrak{A}$ *be an elliptic operator corresponding to the Dirichlet problem (6). Then* $\chi(\mathfrak{A}) = 0$ *in each of the following cases: 1)* n *is even and the operator* A *is differential; 2)* $n > p$.

For $n \le p$ the index of the Dirichlet problem is, in general, different from zero.

3. Deformations in the class of properly elliptic singular integro-differential operators also make it possible to obtain the following result. Let $p = 1$, $s = 2m$, $r = m$. A singular integro-differential boundary operator B is called **fully elliptic** (cf. (^{15})) if A and B are connected by condition (L) on Γ for every properly elliptic operator A.

Theorem 6. *Let* $p = 1$, $s = 2m$, $r = m$; A *be an arbitrary properly elliptic singular integro-differential operator, and* B *a fixed fully elliptic boundary operator. Then the index of the operator* $\mathfrak{A} = (A, B)$ *does not depend on* A.

A special case of Theorem 6 and a consequence of Theorem 5: in the case of a single elliptic equation the index of the Dirichlet problem is equal to 0. This is the theorem from ([16]).

The results of the present note remain essentially valid if G is a smooth manifold with boundary Γ and l is not necessarily an integer. They are partially carried over to elliptic operators acting in stratified spaces (cf. ([14])).

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* These results were obtained before the appearance of article ([14]) by homotopy methods. See ([4]).

Note: Figure translations are in progress. See original paper for figures.

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