MATHEMATICS

A. I. PEROV

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. I. PEROV

ON THE TOPOLOGICAL CHARACTERISTICS OF SOLUTIONS OF MULTIDIMENSIONAL DIFFERENTIAL EQUATIONS

(Presented by Academician L. S. Pontryagin on 17 III 1964)

1. Let E_x be an m-dimensional real space, and E_y an n-dimensional real space. The set of all linear operators acting from E_x to E_y, with the usual definition of the operations of addition and multiplication by real numbers, will be denoted by E_{xy}. In what follows we shall encounter the spaces E_{yy}, $E_{x(yy)}$, and $E_{y(xy)}$.

The null-space and the range of an operator $a \in E_{xy}$ will be denoted respectively by $N(a)$ and $R(a)$. The operator a is called **regular** if it carries out a one-to-one mapping of E_x onto $R(a)$. A subspace $M(a) \subseteq E_x$ will be called a **leading** one if $E_x = N(a) \oplus M(a)$.

Let $A \in E_{y(xy)}$. The operator $\widetilde{A} \in E_{x(yy)}$, uniquely determined by the condition $(Ay)x = (\widetilde{A}x)y$ $(x \in E_x,\ y \in E_y)$, will be called the **adjoint**. We introduce one more operator A_0, acting from $E_x \otimes E_y$ to E_y: $A_0(x \otimes y) = (\widetilde{A}x)y$ (for the terminology and notation see (^{1})).

Let $A, B \in E_{y(xy)}$. We shall say that the operator B is **subordinate** to the operator A, and write $B \subseteq A$, if $R(By) \subseteq R(Ay)$ for all $y \in E_y$. Let us note some simple facts connected with this definition. If $B \subseteq A$, then $R(B_0) \subseteq R(A_0)$; the converse is false. If $R(\widetilde{B}) \subseteq R(\widetilde{A})$, then $B \subseteq A$; the converse is false, except in the cases where either $m = 1$ or $n = 1$.

2. Let $A \in E_{y(xy)}$. Consider the differential equation

$$\frac{dy}{dx} = Ay. \tag{1}$$

It is known (^{2}) that equation (1) has a unique solution satisfying the initial condition

$$y(\xi) = \eta \qquad (\xi \in E_x,\ \eta \in E_y), \tag{2}$$

if the condition

$$\widetilde{A}h\,\widetilde{A}k = \widetilde{A}k\,\widetilde{A}h \qquad (h, k \in E_x) \tag{3}$$

is fulfilled.

Here $\widetilde{A}$ is the operator adjoint to A (see § 1). In what follows we assume condition (3) to be fulfilled. The solution of the Cauchy problem (1)–(2) is written in the form

$$y(x) = \exp(\widetilde{A}(x - \xi))\eta. \tag{4}$$

We denote by $\mathfrak{M}(A;\eta)$ the orbit of the point $\eta \in E_y$ under equation (1). It is not difficult to see that the space E_y decomposes into a collection of nonintersecting orbits. Let us also note that $\overline{\mathfrak{M}}(A;\eta) - \mathfrak{M}(A;\eta)$ consists of orbits of points $y \notin \mathfrak{M}(A;\eta)$ that are limiting for it.

We denote by $y(x;\eta)$ the solution (4) for $\xi = 0$. The function $y(x;\eta)$ is defined on $E_x \oplus E_y$ and has the following properties: 1) $y(0;\eta) = \eta$; 2) $y(x;\eta)$ is continuous in the aggregate of variables; 3) $y(x_1 + x_2;\eta) = y(x_1; y(x_2;\eta))$. Thus, one may say that on E_y there is given a multi-

-dimensional **dynamical system** (for one-dimensional dynamical systems, see (^{3}); multidimensional dynamical systems were studied in a number of works by V. V. Nemytskii, see (^{6})).

Without entering into a detailed exposition of the asymptotic properties of the solutions of equation (1), we note only that a bounded solution is always almost periodic.

3. Let $y(x)$ be a solution of equation (1). Since

$$y(\xi + \Delta x) - y(\xi) = y'(\xi)\Delta x + \varepsilon(\xi, \Delta x), \tag{5}$$

the subspace $R(y'(\xi))$ is tangent to the surface $y = y(x)$ at the point η. In view of the equality $y'(x) = \exp(\widetilde{A}(x-\xi))A\eta$, we obtain that $\dim R(y'(x)) = r$, where r is the rank of the operator $A\eta$. The number r will be called the **dimension** of the solution $y(x)$. Examples can be given in which equation (1) has solutions of all intermediate dimensions.

Let $N = N(A\eta)$, $M = M(A\eta)$; on M the operator $A\eta$ is regular (see item 1).

Theorem 1. *For some positive numbers ρ and ε the inequality*

$$\|y(x_1) - y(x_2)\| \geqslant \varepsilon \|x_1 - x_2\| \tag{6}$$

holds for $\|x_1 - \xi\| \leqslant \rho$, $\|x_2 - \xi\| \leqslant \rho$ and $x_1, x_2 \in \xi + M$.

The image of a neighborhood of the point O in M under the mapping $y(x)$ will be called a neighborhood of the point η in $\mathfrak{M}(A;\eta)$. If the neighborhoods of each point $\zeta \in \mathfrak{M}(A;\eta)$ are defined in this way, then the set $\mathfrak{M}(A;\eta)$ becomes a topological space, for which we shall retain the same notation. It is not difficult to see that the space $\mathfrak{M}(A;\eta)$ is an r-dimensional topological manifold (^{4}).

If $r = n$, then the set $\mathfrak{M}(A;\eta)$ is open in E_y. Let now $0 < r < n$. Introduce in M and in E_y coordinate systems $x^1, \dots, x^r$ and $y^1, \dots, y^n$, respectively. Then the solution $y(x)$ in coordinate form is written as follows:

$$y^i = \varphi^i(x^1, \dots, x^r) \quad (i = 1, \dots, n). \tag{7}$$

Let

$$\frac{D(\varphi^1, \dots, \varphi^r)}{D(x^1, \dots, x^r)}(\xi) \neq 0.$$

Then, solving the first r equations with respect to the variables $x^1, \dots, x^r$ and substituting the resulting expressions into the remaining $(n - r)$ equations, we can write the equation of the surface $y = y(x)$ in the form

$$y^i = \psi^i(y^1, \dots, y^r) \quad (i = r + 1, \dots, n). \tag{8}$$

This fact will be used by us in the proof of the inclusion theorem (see item 6).

4. We now show that the space $\mathfrak{M}(A;\eta)$ can naturally be turned into an (abelian) group. Let $y_1, y_2 \in \mathfrak{M}(A;\eta)$ and $y_1 = \exp(\widetilde{A}x_1)\eta$, $y_2 = \exp(\widetilde{A}x_2)\eta$. Define the sum of the elements y_1 and y_2 by the equality

$$y_1 + y_2 = \exp(\widetilde{A}(x_1 + x_2))\eta. \tag{9}$$

It is not difficult to see that this definition is correct and satisfies all the axioms of an abelian group.

According to what was said above, the mapping $y(x)$ may be regarded as a homomorphic mapping of the group E_x onto the group $\mathfrak{M}(A;\eta)$. Denote by $K(\eta)$ the kernel of this homomorphism. The subgroup $K(\eta)$ can be represented in the form of the direct sum of the group N (see item 3) and of some discrete group S (^{4}).

Let $p = \dim L(S)$, where $L(S)$ is the linear hull of the set S, and $q = r - p$.

Theorem 2. *If, for two sets $\mathfrak{M}(A;\eta)$ and $\mathfrak{M}(B;\zeta)$, the corresponding numbers p and q are equal, then these sets are homeomorphic.*

The converse is true.

For $p = r$ the set $\mathfrak{M}(A;\eta)$ is an r-dimensional torus. For $q = r$ the set $\mathfrak{M}(A;\eta)$ is homeomorphic to r-dimensional space. In the remaining cases $\mathfrak{M}(A;\eta)$ is the topological product of a p-dimensional torus and a q-dimensional space.

Let us also note that the mapping $y(x)$, considered only on M, is a covering mapping of M onto $\mathfrak{M}(A;\eta)$ (^{4}).

5. Let $y(x)$ be a solution of equation (1). Denote by Ω the set of all periods of the function $y(x)$, including also the zero of the space E_x. Recall that a vector $\omega \in E_x$ is called a period of the function $y(x)$ if $y(x+\omega) = y(x)$ for all $x \in E_x$.

It is not difficult to see that $\Omega = N \oplus S$ (see §4); here N is the maximal manifold of constancy for the function $y(x)$, and in S one can specify a basis, consisting of the (fundamental) periods $\omega_1, \dots, \omega_p$, such that any period $\omega \in S$ is representable in the form $\omega = n^1\omega_1 + \dots + n^p\omega_p$, where $n^1, \dots, n^p$ are certain integers.

6. Let us consider an example. Let $m = n = 2$. If in E_x and in E_y coordinate systems x^1, x^2 and y^1, y^2 are chosen, then equation (1) is written in the form of a system of partial differential equations

$$\frac{\partial y^i}{\partial x^j} = a^i_{j1} y^1 + a^i_{j2} y^2 \quad (i, j = 1, 2), \tag{10}$$

and the initial condition (2) takes the form

$$y^i(\xi^1, \xi^2) = \eta^i \quad (i = 1, 2). \tag{11}$$

Condition (3) imposes the following restrictions on the coefficients of the system:

$$\begin{vmatrix} a^1_{12} & a^1_{22} \\ a^2_{11} & a^2_{21} \end{vmatrix} = \begin{vmatrix} a^1_{11} & a^1_{12} \\ a^1_{21} & a^1_{22} \end{vmatrix} + \begin{vmatrix} a^1_{12} & a^1_{22} \\ a^2_{12} & a^2_{22} \end{vmatrix} = 0. \tag{12}$$

One can show that, by a linear change of the variables y^1, y^2, it is always possible to arrange that in the new coordinate system the matrix of the operator $\widetilde{A}$ will have one of the following canonical forms:

$$\begin{pmatrix} \lambda(x) & 0 \\ 0 & \mu(x) \end{pmatrix}, \quad \begin{pmatrix} \alpha(x) & \beta(x) \\ -\beta(x) & \alpha(x) \end{pmatrix}, \quad \begin{pmatrix} \lambda(x) & \alpha(x) \\ 0 & \lambda(x) \end{pmatrix}, \tag{13}$$

where the coefficients of the written matrices are linear functionals.

It follows from this fact that the dimension of almost all solutions of system (10) coincides with the dimension of the linear hull of the functionals λ and μ in the first case, α and β in the second case, and λ and α in the third case, i.e. with the rank of the operator $\widetilde{A}$.

In the second of the cases listed above, if α and β are linearly independent, then $p = q = 1$ for any nonzero solution of system (10). System (10) was studied in (^{5}) using the theory of hypercomplex systems.

7. The following is closely related to the circle of questions under study.

Theorem 3 (embedding theorem). *Let A and B be two operators from $E_{y(x)}$, each of which satisfies condition (3).*

Then the inclusion $\mathfrak{M}(B;\eta) \subseteq \mathfrak{M}(A;\eta)$ holds for all $\eta \in E_y$ if and only if $B \subseteq A$.

The proof of this theorem, in the part concerning necessity, is based on the following lemma.

Lemma. Let $z(t) \in C^1(\Delta)$ and $z(t) \in \mathfrak{M}(A;\eta)$.

Then $\dot{z}(t) \in R(Az(t))$, and if $z(t) = y(x(t))$, where $x(t) \in C(\Delta)$, then $x(t) \in C^1(\Delta)$ and

$$\dot{x}(t) = \left\{ \frac{dy(x(t))}{dx} \right\}^{-1} \dot{z}(t). \tag{14}$$

Voronezh
State University

Received
13 III 1964

References

1. P. Halmos, *Finite-Dimensional Vector Spaces*, Moscow, 1963.
2. M. K. Gavurin, Uchen. zap. LGU, ser. matem., issue 19 (1950).
3. V. V. Stepanov, V. V. Nemytskii, *Qualitative Theory of Differential Equations*, Moscow-Leningrad, 1949.
4. L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.
5. H. Egli, Comm. math. Helv., **36**, Nos. 2, 3 (1961).
6. *Mathematics in the USSR over 40 Years*, Moscow, 2, 1959.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.