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Abstract

Full Text

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On φ-Complete Spaces Metrized over a Semifield

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In note (3), for a space metrized over a semifield, the notion of φ-completeness was introduced and it was proved that every φ-complete space is a space of the second category in itself. A slight modification of the proof makes it possible, from the same assumptions, to obtain a stronger conclusion. The present note is devoted to the exposition of this question.

Let us first recall the notion of a metric space over a semifield. Let E be some semifield and K_E the set of all its positive elements. A set X is called a **metric space over the semifield** E if a mapping (called a metric) is given:

$$\rho : X \times X \to K_E,$$

satisfying the following conditions:

1. $\rho(x, y) = 0$ if and only if $x = y$.
2. $\rho(x, y) = \rho(y, x)$.
3. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

If U is an arbitrary neighborhood of zero in the metrizing semifield E and $x \in X$, then by $\Omega(x, U)$ we shall denote the set of all those elements $y \in X$ for which $\rho(x, y) \in U$. The collection of all sets of the form $\Omega(x, U)$ may be taken as a base of neighborhoods in X (see (1), p. II, 1). The topology obtained in this way is called the **natural topology of the metric space** X. A topological space X is called **metrized over a semifield** if a metric is introduced in it and the natural topology of the metric space X coincides with the original topology of the space X.

Let us recall the basic definitions of (3). Let Δ be some infinite set; Ω the set of all its finite subsets, partially ordered by inclusion; φ a mapping of the set $\Omega \setminus \Delta$ into Ω, satisfying the following conditions: 1) $\varphi(\xi) < \xi$ for all $\xi \in \Omega \setminus \Delta$; 2) whatever $\xi_1, \xi_2 \in \Omega$ may be, there is an element $\xi_3 \in \Omega$ and a natural number k such that the relations $\xi_3 > \xi_2$ and $\varphi^k(\xi_3) = \xi_1$ hold. The proof of existence and the construction of such a mapping φ are given in (2).

Definition 1. We shall say that ξ_1 is a φ**-larger element** than ξ_2, and denote this by

$$\xi_1 \overset{\varphi}{>} \xi_2,$$

if there exists a natural number k such that $\varphi^k(\xi_1) = \xi_2$.

Definition 2. A sequence of type Ω of points of a topological space X, metrized over R_Δ, will be called φ-**fundamental** if for every neighborhood of zero U in R_Δ there exists $\xi_0 \in \Omega$ such that from

$$\xi_1 > \xi_0, \qquad \xi_2 \overset{\varphi}{>} \xi_1$$

it follows that $\rho(x_{\xi_1}, x_{\xi_2}) \in U$.

Definition 3. A point $a \in X$ will be called a φ-**limit** for a sequence of type Ω if for every neighborhood of zero U in R_Δ and every $\xi_0 \in \Omega$ there exists an element

$$\xi \overset{\varphi}{>} \xi_0,$$

such that $\rho(x_\xi, a) \in U$.

Definition 4. We shall call the space X φ-complete if, for every φ-fundamental sequence $\{x_\xi\}$, there exists at least one φ-limit point.

Theorem 1. *Let the space X be metrized over R_Δ, and suppose that for some mapping φ satisfying the conditions 1) and 2) mentioned above, the space X is φ-complete. Then the space X cannot be represented as the sum of m nowhere dense subsets, where m is the cardinality of the set Δ (or, what is the same thing, the intersection of open everywhere dense subsets of the space X is nonempty).*

Proof. Let $\xi = \{q_1, \dots, q_n\}$ be an arbitrary element of the set Ω. By U_ξ we shall denote the neighborhood of zero in R_Δ consisting of functions which take, at $q_i \in \xi$, values whose moduli are less than $2^{-|\xi|}$, where $|\xi|$ is the number of elements in ξ. Let $\{G_q\}$ be an arbitrary family of open everywhere dense subsets in X (the index ranges over the set Δ).

We construct, by induction on ξ, such a φ-fundamental sequence $\{x_\xi\}$ that

$$x_\xi \in \bigcap_{q \in \xi} G_q.$$

Simultaneously by induction we construct a collection of such neighborhoods V_ξ that $x_\xi \in V_\xi$, $x_{\varphi(\xi)} \in V(\xi)$, and

$$\overline{V}(\xi) \subset V_{\varphi(\xi)} \cap \left(\bigcap_{q \in \varphi(\xi)} G_q \right).$$

If $\xi = q$, then as x_ξ we take any point of the set G_q. For V_ξ for these ξ we take G_q.

Suppose now that all x_ξ and V_ξ for which $|\xi| < k$, where $k > 1$, have been constructed, and let $|\xi_0| = k$. Since $\varphi(\xi_0) < \xi_0$, we have $|\varphi(\xi_0)| < k$, and hence, by the induction hypothesis, the point $x_{\varphi(\xi_0)}$ has already been constructed.

Take such a neighborhood V_{ξ_0} of the point $x_{\varphi(\xi_0)}$ that

$$\overline{V}_{\xi_0} \subset V_{\varphi(\xi_0)} \cap \left(\bigcap_{q \in \varphi(\xi_0)} G_q \right) \cap \Omega(x_{\varphi(\xi_0)}, U_{\xi_0}).$$

The intersection of the constructed neighborhood V_{ξ_0} with $\bigcap_{q \in \xi_0} G_q$ is nonempty, since the sets G_q are everywhere dense in X. Take an arbitrary point in the intersection

$$V_{\xi_0} \cap \left(\bigcap_{q \in \xi_0} G_q \right)$$

and denote it by x_{ξ_0}. The induction carried out makes it possible to construct x_ξ and V_ξ for all $\xi \in \Omega$.

We prove that the sequence obtained is φ-fundamental. Let U be some neighborhood of zero in R_Δ. There exists an element $\xi_0 \in \Omega$ such that $U_{\xi_0} \subset U$ (see (1)). Let now $\xi_1 > \xi_0$, $\xi_2 \overset{\varphi}{>} \xi_1$, and let k be such a natural number that $\varphi^k(\xi_2) = \xi_1$. Then

$$\rho(x_{\xi_2}, x_{\xi_1}) \leq \rho(x_{\xi_2}, x_{\varphi(\xi_2)}) + \rho(x_{\varphi(\xi_2)}, x_{\varphi^2(\xi_2)}) + \cdots + \rho(x_{\varphi^{k-1}(\xi_2)}, x_{\xi_1}).$$

But

$$\rho(x_{\varphi^i(\xi_2)}, x_{\varphi^{i+1}(\xi_2)}) \in U_{\varphi^i(\xi_2)} \subset \frac{1}{2} U_{\varphi^{i+1}(\xi_1)} \subset \cdots \subset \frac{1}{2^{k-i}} U_{\varphi^k(\xi_2)} = \frac{1}{2^{k-i}} U_{\xi_1}$$

(recall that $|\varphi(\xi)| \leq |\xi| - 1$). Thus, $\rho(x_{\varphi^i(\xi_2)}, x_{\varphi^{i+1}(\xi_2)})$ at $q_l \in \xi_1$ assumes values less than

$$\frac{1}{2^{k-i}} \frac{1}{2^{|\xi_1|}},$$

and consequently $\rho(x_{\xi_2}, x_{\xi_1})$ assumes at these q_l values less than

$$\frac{1}{2^{|\xi_1|}} \sum_{i=0}^{k-1} \frac{1}{2^{k-1}} < \frac{1}{2^{|\xi_1|}}.$$

Hence,

$$\rho(x_{\xi_2}, x_{\xi_1}) \in U_{\xi_1} \subset U_{\xi_0}.$$

Thus, the sequence $\{x_\xi\}$ is φ-fundamental. By virtue of the φ-completeness of the space X, it has a φ-limit point a. Let U be an arbitrary neighborhood of zero in R_Δ. For any $\xi_0 \in \Omega$ there is an element ξ such that $\xi \overset{\varphi}{>} \xi_0$ and $\rho(x_\xi, a) \in U$. But by the construction of the sequence $\{x_\xi\}$ and the fact that $\xi \overset{\varphi}{<} \xi_0$, we have

$$x_\xi \in V_\xi \subset V_{\xi_0},$$

and therefore

$$a \in V_{\xi_0} \subset \bigcap_{q \in \varphi(\xi_0)} G_q.$$

Since this is true for any element $\xi_0 \in \Omega$, and since for any element $q \in \Delta$ there exists such a ξ_0 that $q \in \varphi(\xi_0)$ (see property 2 of the mapping φ), it follows that $a \in \bigcap_{q \in \Delta} G_q$, as was required to prove.

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Note: Figure translations are in progress. See original paper for figures.