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Abstract

Full Text

MATHEMATICS

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LOCAL SATURATION OF A FAMILY OF LINEAR POSITIVE OPERATORS

(Presented by Academician V. I. Smirnov, 15 I 1964)

I. Let $\mathcal{L}_\lambda(f;x)$ be a family of linear positive operators (l.p.o.), whose domain of definition H, in particular, contains also the set of all functions $f(x)$ having on $[a,b]$ an absolutely continuous $(2m-1)$-st derivative $f^{(2m-1)}(x)$. Define the linear operator

$$M_\lambda(f;x)=\frac{1}{\tau_\lambda^{[2m]}}\left[\mathcal{L}_\lambda(f;x)-\sum_{k=0}^{2m-1}\frac{f^{(k)}(x)}{k!}\tau_\lambda^{[k]}\right],$$

where $\tau_\lambda^{[k]}=\mathcal{L}_\lambda[(t-x)^k;x]$ and $m\geq 1$ is a given integer.

Denote by $H(n,x)$ the set of all functions $f(x)$, defined on $[a,b]$ and having a finite derivative of order n, $f^{(n)}(x)$, at the point $x\in[a,b]$.

Let the operator $M_\lambda(f;x)$ map every function $f(x)\in H$ into a function continuous on $[a,b]$. Suppose, moreover, that the conditions

$$|M_\lambda(f;x)|\leq M<+\infty, \tag{1}$$

$$\lim_{\lambda\to\infty}M_\lambda(f;x)=\frac{f^{(2m)}(x)}{(2m)!} \tag{2}$$

are satisfied for every function $f\in H(2m,x)$ on the set of all points x of the interval $[a,b]$ where the finite $f^{(2m)}(x)$ exists. It is known $(^{3,4})$ that, in order that the asymptotic equality (2) hold for all $f\in H(2m,x)$, it is necessary and sufficient that the condition

$$\lim_{\lambda\to\infty}\frac{\tau_\lambda^{[2m+2j]}}{\tau_\lambda^{[2m]}}=0$$

hold for at least one value $j=1,2,\ldots$.

In what follows we also assume that for the l.p.o. the condition

$$\lim_{\lambda\to\infty}\int_a^b [M_\lambda(f;x)\varphi(x) - M_\lambda(\varphi;x)f(x)]\,dx = 0 \tag{3}$$

is satisfied for all $f(x) \in H$ and $\varphi(x) \in H$.

Theorem 1. If $M_\lambda(f;x) = o(1)$ as $\lambda \to \infty$ uniformly on $[a,b]$, then $f(x)$ is an algebraic polynomial of degree $\leq 2m-1$.

Proof. Denote by $C^{(2m)}(a,b)$ the totality of all functions $f(t)$, defined on the interval $[a,b]$, having on this interval a continuous derivative of order $2m$ and equal to zero outside $[a,b]$. Since $\lim_{\lambda\to\infty} M_\lambda(f;x) = 0$ uniformly on $[a,b]$, we have

$$\lim_{\lambda\to\infty}\int_a^b M_\lambda(f;x)\varphi(x)\,dx = 0 \tag{4}$$

for every function $\varphi \in C^{(2m)}(a,b)$.

We note that, by virtue of (3), the relation

$$\lim_{\lambda\to\infty}\int_a^b M_\lambda(f;x)\varphi(x)\,dx = \lim_{\lambda\to\infty}\int_a^b M_\lambda(\varphi;x)f(x)\,dx. \tag{5}$$

holds.

On the other hand, the equality

$$\lim_{\lambda\to\infty} M_\lambda(\varphi;x) = \frac{\varphi^{(2m)}(x)}{(2m)!}$$

is valid on $[a,b]$ for $\varphi \in C^{(2m)}(a,b)$. Consequently, from (1), (4), and (5) we find

$$\int_a^b \varphi^{(2m)}(x)f(x)\,dx = 0$$

for every function $\varphi \in C^{(2m)}(a,b)$. Hence, on the basis of the fundamental lemma of the calculus of variations, it follows that $f(x)$ is an algebraic polynomial of degree $2m-1$.

Theorem 2. *In order that, as $\lambda \to \infty$, the relation*

$$M_\lambda(f;x) = O(1) \tag{6}$$

hold uniformly on $[a,b]$, it is necessary and sufficient that $f^{(2m-1)}(x) \in \mathrm{Lip}\,1$.

Necessity. Since (6) holds uniformly on $[a,b]$, by the weak compactness of the functionals of the space $L_1(a,b)$, there exist a subsequence $\{\lambda_k\}$ and a function $\psi(x) \in L_\infty(a,b)$ such that

$$\lim_{k\to\infty} \int_a^b M_{\lambda_k}(f;x)\varphi(x)\,dx = \int_a^b \psi(x)\varphi(x)\,dx \tag{7}$$

for every $\varphi \in C^{(2m)}(a,b)$. The left-hand side of relation (7), by virtue of (1), (2), and (3), is equal to

$$\lim_{k\to\infty} \int_a^b M_{\lambda_k}(f;x)\varphi(x)\,dx = \lim_{k\to\infty} \int_a^b M_{\lambda_k}(\varphi;x)f(x)\,dx = \frac{1}{(2m)!} \int_a^b \varphi^{(2m)}(x)f(x)\,dx. \tag{8}$$

The right-hand side of equality (7), in turn, can be represented in the form

$$\int_a^b \psi(x)\varphi(x)\,dx = \int_a^b \varphi^{(2m)}(x)F_{2m}(x)\,dx, \tag{9}$$

where $F_{2m}(x)$ is the $2m$-th indefinite integral of $\psi(x)$.

Consequently, from (7), (8), and (9) we find

$$\int_a^b \varphi^{(2m)}(x)\left[f(x) - (2m)!F_{2m}(x)\right]dx = 0$$

for every $\varphi \in C^{(2m)}(a,b)$. Hence we conclude that $f(x) - (2m)!F_{2m}(x)$ is an algebraic polynomial of degree $2m-1$, i.e. $f^{(2m-1)}(x) \in \operatorname{Lip} 1$.

Sufficiency. If $f^{(2m-1)}(x) \in \operatorname{Lip} 1$, then $f^{(2m)}(x)$ exists almost everywhere and is bounded on $[a,b]$. Hence, by virtue of (1),

$$\|M_\lambda(f;x)\|_{L_\infty} = O(1)$$

as $\lambda \to \infty$. Owing to the continuity of $M_\lambda(f;x)$, from the last relation we find that (6) holds uniformly on $[a,b]$.

Theorems 1 and 2 determine the orders and classes of local saturation (see, for this, (5)) of the family $\mathcal{L}_\lambda(f;x)$.

II. By analogous reasoning one determines the classes and orders of local saturation of the family of l.p.o. $W_\lambda(f;x)$, defined on the set of 2π-periodic functions. We shall show this for $m=1$; the case $m>1$ is treated similarly. We assume that for the family $W_\lambda(f;x)$ the conditions stated at the beginning are also satisfied, in particular conditions (1), (2), and (3), in which $M_\lambda(f;x)$ must be replaced by the operator

$$N_\lambda(f;x) = \frac{1}{\mu_\lambda^{[2]}}\,[W_\lambda(f;x) - f(x) - f'(x)\mu_\lambda^{[1]}],$$

where

$$\mu_\lambda^{[k]} = W_\lambda\left[2^k \sin^k \frac{t-x}{2}; x\right].$$

Theorem 3. 1. If, as $\lambda \to \infty$,

$$N_\lambda(f;x) = o(1) \tag{10}$$

uniformly, then $f(x)$ is a constant.

2. In order that, as $\lambda \to \infty$, the relation

$$N_\lambda(f;x) = O(1) \tag{11}$$

hold uniformly, it is necessary and sufficient that $f'(x) \in \operatorname{Lip} 1$.

Proof. 1. Since (10) holds uniformly on $[-\pi,\pi]$, we have

$$\lim_{\lambda\to\infty} \int_{-\pi}^{\pi} N_\lambda(f;x)e^{-ikx}\,dx = 0.$$

Using condition (3) and the asymptotic equality (2), we find

$$\lim_{\lambda\to\infty} \int_{-\pi}^{\pi} N_\lambda(f;x)e^{-ikx}\,dx = \lim_{\lambda\to\infty} \int_{-\pi}^{\pi} N_\lambda(e^{-ikt};x)f(x)\,dx = -\pi k^2 C_k(f) = 0$$

for all k, where $C_k(f)$ is the k-th Fourier coefficient of $f(x)$. Consequently, $C_k(f) = 0$ for $k = \pm1, \pm2, \ldots$, i.e. $f(x) = \text{const}$.

2. Let us prove necessity. Taking into account that (11) holds uniformly on $[-\pi,\pi]$, by the weak compactness of the space $L_1(-\pi,\pi)$ one can find a function $\psi(x) \in L_\infty(-\pi,\pi)$ and a subsequence λ_k such that the relation

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} N_{\lambda_k}(f;x)e^{-ikx}\,dx = \int_{-\pi}^{\pi} \psi(x)e^{-ikx}\,dx$$

is valid.

The left-hand side is equal to $-\pi k^2 C_k(f)$. Consequently,

$$-\pi k^2 C_k(f) = \int_{-\pi}^{\pi} \psi(x) e^{-ikx}\, dx$$

for all k. Hence it follows that $f''(x) \in L_\infty(-\pi, \pi)$, i.e. $f'(x) \in \mathrm{Lip}\, 1$.

The proof of sufficiency is obvious.

As an example, consider the l.p.o. of P. P. Korovkin (^{1})

$$A_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) V_n(t)\, dt, \tag{12}$$

where

$$V_n(t) = \frac{1}{2} \sum_{k=1}^{n} \rho_k^{(n)} \cos kt, \qquad \rho_1^{(n)} = \cos \frac{\pi}{n+2}, \ldots$$

For the l.p.o. (12) all the conditions of Theorem 3 are satisfied, and moreover

$$\mu_n^{[1]} = 0, \qquad \mu_n^{[2]} = 2[1 - \rho_1^{(n)}] = 2\left(1 - \cos\frac{\pi}{n+2}\right) \simeq \frac{\pi^2}{n^2} \quad (n \to \infty).$$

Thus, from Theorem 3 we obtain

Corollary. 1. If, as $n \to \infty$, $A_n(f; x) - f(x) = o(n^{-2})$ uniformly, then $f(x) =$ const.

2. In order that, as $n \to \infty$, the relation $A_n(f; x) - f(x) = O(n^{-2})$ hold uniformly, it is necessary and sufficient that $f'(x) \in \mathrm{Lip}\, 1$.

The last result is customarily written as follows:

$$\mathrm{Sat}[A_n]_C = [\{f \mid f' \in \mathrm{Lip}\, 1\},\ n^{-2},\ \text{constant}].$$

For another l.p.o. of P. P. Korovkin $^{(2)}$,

$$A_r(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) u_r(t)\, dt,$$

$$u_r(t) = \frac{1}{2} + \sum_{k=1}^{\infty} r^{k^2} \cos kt,$$

we have

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$$\mathrm{Sat}[A_n]_C = [\{f \mid f' \in \mathrm{Lip}\,1\},\ 1-r,\ \text{constant}].$$

On the basis of Theorem 3 one can also verify that the class and order of saturation of the well-known l.p.o. of Vallée-Poussin and Jackson (see $^{(6)}$) are determined in the form

$$\mathrm{Sat}[B_n]_C = [\{f \mid f' \in \mathrm{Lip}\,1\},\ n^{-1},\ \text{constant}],$$

$$\mathrm{Sat}[D_n]_C = [\{f \mid f' \in \mathrm{Lip}\,1\},\ n^{-2},\ \text{constant}],$$

respectively. These results were obtained earlier by A. Kh. Turetskii $^{(7)}$.

We note that assertions analogous to Theorems 1-3 are also valid in the metric of the space L_p $(p > 1)$. Moreover, if approximation of functions by means of l.p.o. on the unbounded interval $(-\infty, \infty)$ is considered, then in proving the corresponding assertions one should use the Fourier transform of functions instead of Fourier coefficients.

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