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Abstract

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ON STABLE GROUPS OF AUTOMORPHISMS

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In our paper ([1]) the following results were obtained:

In an arbitrary group G, the Baer nilradical $N(G)$ ([2]) coincides with the set of all outer nilpotent elements of the group G (([1]), Theorem 1).

If G is an arbitrary group and Φ is its outer nilpotent group of automorphisms, then $[G, \Phi] \subseteq N(G)$ (([1]), Corollary 1 to Theorem 3).

In the present note, with the aid of the quasinilpotent radical $K(G)$ ([3]), analogous results are obtained, which are of interest in connection with the problem of constructing general structural theories of groups of automorphisms.

Let us recall some definitions and notation (most of them are taken from ([4], [5])).

Let G be an arbitrary group, and Φ its group of automorphisms. Φ is called **stable** if in G there is an ascending (generally speaking, transfinite) normal series of Φ-admissible subgroups on whose factors the elements of Φ induce identical automorphisms. The corresponding normal series in G is called **stable relative to** Φ. By $\hat{g}$ we denote the inner automorphism induced by the element $g \in G$ in G. An element g of any group G will be called **stable** if the inner automorphism $\hat{g}x = g^{-1}xg$ is stable relative to the group G. A stable subset of the group G is defined analogously.

Let G be an arbitrary group, Φ its group of automorphisms, and let $g \in G$, $\sigma \in \Phi$. The element $g^{-1}\sigma g$ is called the σ-**commutator** of the element g and is denoted by $[g, \sigma]$, i.e. $[g, \sigma] = g^{-1}\sigma g$. The subgroup of the group G generated by all commutators of the form $[g, \sigma]$, $g \in G$, $\sigma \in \Phi$, is called the **mutual commutant** of G and Φ and is denoted by $[G, \Phi]$.

For groups of automorphisms Φ of the group G, define inductively:

$$G_0(\Phi) = G, \quad G_1(\Phi) = [G, \Phi], \dots, \quad G_\alpha(\Phi) = [G_{\alpha-1}(\Phi), \Phi], \dots,$$

and if α is a limit ordinal, then

$$G_\beta(\Phi) = \bigcap_{\beta<\alpha} G_\alpha(\Phi).$$

It is easily proved that this descending series of Φ-commutants is normal, and Φ is outer nilpotent if and only if this series terminates ([9]).

For an element $g \in G$, define inductively

$$g_0 = g, \quad g_1 = [g, \sigma], \dots, \quad g_n = [g_{n-1}, \sigma], \dots$$

An automorphism σ is called a **nilautomorphism** if for every $g \in G$ there exists an $n = n(g, \sigma)$ such that $g_{n+1} = [g_n, \sigma] = 1$. It is easy to see that if Φ is a stable group relative to G, then all elements of Φ will be nilautomorphisms.

We call a group G **quasinilpotent** ([3]) if every finite subset of its elements generates a subinvariant subgroup of the group G. The subgroup of any group G generated by all invariant quasinilpotent subgroups of the group G is called the **quasinilpotent radical** of the group G and is denoted by $k(G)$. In ([3]) it is shown by examples that in the general case the nilradical

Baer' s $N(G)$ lies strictly in the quasinilpotent radical $k(G)$, and the quasinilpotent radical lies strictly in the locally nilpotent radical $R(G)$ of B. I. Plotkin ([6]), i.e.

$$N(G) \subset k(G) \subset R(G).$$

Lemma. Let G be an arbitrary group and $g \in G$. The element g belongs to the quasinilpotent radical $k(G)$ if and only if it is stable with respect to G.

Proof. Let $g \in k(G)$. By the property of the quasinilpotent radical ([3], Theorem 5), the cyclic subgroup $\{g\}$ is subinvariant in G, i.e. in the group G there is an ascending normal series whose first term is the cyclic subgroup $\{g\}$. It is easy to verify that g is a stable element with respect to G. Conversely, let g be a stable element with respect to the group G, i.e. the inner automorphism $\hat{g}x = g^{-1}xg$ is stable in G. Then, by assumption, the group G has a normal series

$$E \subset G_1 \subset G_2 \subset \cdots \subset G_\alpha \subset \cdots \subset G_\gamma = G, \tag{1}$$

all subgroups G_α of which are $\hat{g}$-admissible, i.e.

$$\hat{g}G_\alpha = g^{-1}G_\alpha g = G_\alpha,$$

and for every α we have $[G_{\alpha+1}, g] \subseteq G_\alpha$. Multiply every term of the normal series (1) by the cyclic subgroup $\{g\}$. We obtain a new normal series whose first term is the subgroup $\{g\}$. By the property of the quasinilpotent radical ([3]), we obtain $g \in k(G)$. The lemma is proved.

From this lemma, using the property of the quasinilpotent radical, we obtain the following result.

Theorem 1. In an arbitrary group G, the quasinilpotent radical coincides with the set of all stable elements.

See the analogous Theorem 2 in ([8]).

Corollary 1. A group G is quasinilpotent if and only if all its elements are stable.

Corollary 2. If G is an M-group, i.e. G is a group with the maximal condition for subgroups, and all its elements are stable, then the group itself is nilpotent.

Theorem 2. Let G be an arbitrary group and $A = (a_1, a_2, \dots, a_n)$ a finite subset of the group G. The subgroup $\{A\}$ belongs to the quasinilpotent radical if and only if the set A is stable with respect to G.

For the proof of this theorem it suffices to use the known fact ([3]): a subgroup H of any group G generated by a finite number of subinvariant cyclic subgroups of the group G lies in the quasinilpotent radical of the group G.

Corollary 1. In an RN^*-group G, the set of all elements is a subgroup, and this subgroup coincides with the radical $k(G)$.

For this it suffices to note that in an RN^*-group $k(G) = R(G)$.

Theorem 3. Let G be an arbitrary group and Φ its stable group of automorphisms; then

$$[G, \Phi] \subseteq k(G).$$

Proof. As is known ([7], pp. 81–82), the holomorph Γ of the group G can be represented as the product of the group G and the automorphism group of G. The group G is contained in its holomorph $\Gamma = G\Phi$ as a normal divisor, and all automorphisms of the group arise from inner automorphisms of the holomorph Γ.

Denote by $\overline{G} = \{G, \overline{\sigma}\}$ the subgroup in the holomorph Γ of the group G generated by the group G and the element $\overline{\sigma}$ corresponding to the stable automorphism $\sigma \in \Phi$. We shall show that $\overline{\sigma}$ is a stable element in $\overline{G}$.

Since σ is a stable automorphism of the group G, the group G has a σ-stable normal series (1). Now multiply all terms of the normal series (1) by the cyclic subgroup $\{\sigma\}$. We obtain an ascending normal series of the group $\overline{G}$, whose first term is the cyclic subgroup $\{\sigma\}$. From the lemma it follows that $\overline{\sigma} \in k(\overline{G})$. Since the quasinilpotent-

the radical $k(\overline{G})$ is a normal divisor of the group $\overline{G}$, for any element $g \in G$ we have $[g, \overline{\sigma}] \in k(\overline{G})$; on the other hand:

$$\sigma g = \overline{\sigma}^{-1} g \overline{\sigma}, \qquad g^{-1}\sigma g = g^{-1}\overline{\sigma}^{-1} g \overline{\sigma},$$

i.e. $[g, \sigma] = [g, \overline{\sigma}] \in G$.

Since G is invariant in $\overline{G}$, we have $k(G) \subseteq k(\overline{G})$. Hence $k(G) \subseteq G \cap k(\overline{G})$. But since $G \cap k(\overline{G}) \subseteq k(G)$, we obtain

$$k(G) = G \cap k(\overline{G}).$$

Consequently, $[g, \sigma] \in k(G)$, where $g \in G,\ \sigma \in \Phi$. The theorem is proved.

Corollary 1. Let G be an arbitrary group and let Φ be its stable group of automorphisms; then

$$[G, \Phi] \subseteq RN^*(G).$$

This is obvious, since the quasi-nilpotent radical $k(G)$ of the group G lies in the $RN^*(G)$-radical of this group ([3]).

With the aid of this theorem and the results of the paper ([1]), which were cited at the beginning of this article, some previously known results are obtained as corollaries. We note several of them.

Corollary 2. If G is a group and Φ its stable group of automorphisms, then

$$[G, \sigma] \subseteq R(G)$$

(([4]), corollary to Theorem 7).

This is obvious, since under the conditions of the theorem $[G, \sigma] \subseteq k(G)$ and $k(G) \subseteq R(G)$.

Corollary 3. Let G be an M-group, and let Φ be its stable group of automorphisms; then the group Φ is nilpotent (([8]), Theorem 1).

This follows easily, since under the conditions of the theorem $[G, \Phi]$ is a nilpotent group.

Corollary 4. If in the group G there is a finite invariant series stable with respect to its group of automorphisms Φ, then the group Φ is nilpotent (see ([9])).

Using Theorem 2 and Lemma 4.1 from ([5]), we obtain

Theorem 4. Let G be an arbitrary group, and let Φ be its group of automorphisms. Then the set of all stable automorphisms of the group G that lie in $R(\Phi)$ is a normal divisor in Φ.

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CITED LITERATURE

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