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Abstract

Full Text

MATHEMATICS

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ON THE FORMULATION OF SOME INVERSE PROBLEMS

In solving various problems of engineering, it is very often necessary to reconstruct the principal quantitative characteristics of the phenomenon or process being studied, if the complete mathematical formulation of the problem is known and, consequently, a fundamental description is given of the operator L, of the coefficients of the equation, and of the class of functions to which the solution belongs ([1–3]).

It is assumed that we do not possess complete information about the quantitative characteristics of the coefficients of the equations characterizing the operator L. The problem consists in reconstructing the operator L from a given set of certain functionals of the problems. As the functionals we shall take the readings of physical instruments. In what follows we restrict ourselves to the study of linear problems and linear functionals.

Consider the equation in operator form

$$L\varphi = f, \tag{1}$$

where $\varphi \in \Phi,\ f \in F$, and $\Phi \subset F_R,\ F \subset F_R$.

In the space F_R we define the scalar product (g, h), where g and h belong to the space F_R or to a part of it. We shall further introduce into consideration the adjoint operator L^*, in the sense of Lagrange, associated with L by the relation

$$(h, Lg) = (g, L^*h). \tag{2}$$

For simplicity we shall assume that the operator L and the functions g, h are real; $g \in \Phi,\ h \in \Phi^*$.

Consider a certain linear functional $J_p(\varphi)$, defined by the scalar product

$$J_p(\varphi) = (p, \varphi). \tag{3}$$

If the solution of equation (1) is known, then formula (3) makes it possible to compute any linear functional. Let us now fix the functional $J_p(\varphi)$ and formulate the adjoint problem:

$$L^* \varphi_p^* = p, \tag{4}$$

where $\varphi_p^* \in \Phi^*,\ p \in P$, and $\Phi^* \subset F_R,\ P \subset F_R$.

Multiply equation (1) scalarly by φ_p^*, and equation (4) by φ, and subtract the result of one from the other. Then we obtain

$$(\varphi_p^*, L\varphi) - (\varphi, L^* \varphi_p^*) = (f, \varphi_p^*) - (p, \varphi). \tag{5}$$

The left-hand side of equality (5) is equal to zero; therefore we shall have

$$(f, \varphi_p^*) = (p, \varphi). \tag{6}$$

Taking relation (3) into account, we may write

$$J_p(\varphi) = (f, \varphi_p^*). \tag{7}$$

Thus, the functional $J_p(\varphi)$ can be obtained either with the aid of the solution of the basic equation (1), or with the aid of the adjoint equation (4).

It should be noted that the indicated approach to computing the functional $J_p(\varphi)$ with the aid of the solution of the adjoint problem has essentially been used in many areas of science and technology. This approach was substantially developed in physics in connection with the creation of the theory of nuclear reactors and the development of methods for calculating radiation shielding ([4–6]). From a more general point of view, the method of adjoint equations is considered in ([6]). Further development of the results of this work was given in ([7–9]), etc. The results of the investigations carried out in ([6]) will serve as the basis for our formulation of inverse problems in the sense indicated above.

Suppose that the solution of equation (1) and the solution of equation (4) have been found for some prognostic choice of the sought parameters of the operator L. Suppose further that the indicated parameters uniquely determine the operator L and the corresponding operator L^*. Let $\{\alpha_i\}$ be the set of sought parameters of the problem, and $\{\alpha_i^{(1)}\}$ their prognostic values. Then, analogously to (1) and (4), we have the following equations:

$$L_1 \varphi_1 = f_1, \tag{8}$$

$$L_1^* \varphi_{1p}^* = p. \tag{9}$$

Since the solutions of equations (8) and (9) are known, it is possible to compute the values of the functionals $J_p(\varphi_1)$. If the parameters $\{\alpha_i\}$ have been chosen correctly, all possible functionals $J_p(\varphi_1)$ coincide with the readings of the corresponding instruments and, consequently, the prognostic values $\{\alpha_i^{(1)}\}$ will be the sought solutions of the inverse problem. However, such a case is ideal. In reality the functionals $J_p(\varphi_1)$ will not coincide with $J_p(\varphi)$, so that

$$J_p(\varphi) - J_p(\varphi_1) = \delta J_p, \tag{10}$$

where δJ_p is the variation of the functional, which is the discrepancy between the measured and calculated values of J_p.

The method of trial selection of the parameters $\{\alpha_i\}$ could be continued, but it does not lead to an effective algorithm because of the large amount of computation. An effective solution of the inverse problem may be hoped for by using the formulas of perturbation theory for the functional J_p.

Along with the prognostic equations (8), (9), consider the exact equation (1). We shall assume that the following representations hold:

$$\begin{aligned} L &= L_1 + \delta L, \\ f &= f_1 + \delta f, \\ \alpha_i &= \alpha_i^{(1)} + \delta\alpha_i. \end{aligned} \tag{11}$$

We multiply equation (1) scalarly by φ_{1p}^*, and equation (9) by φ, and subtract the results. Then we obtain

$$(\varphi_{1p}^*, L\varphi) - (\varphi, L_1^*\varphi_{1p}^*) = (\varphi_{1p}^*, f) - (\varphi, p). \tag{12}$$

Using relations (9), (10), and (11), we arrive at the formula of perturbation theory for functionals

$$(\varphi_{1p}^*, \delta L\varphi - \delta f) = -\delta J_p. \tag{13}$$

If at our disposal there is a set of n independent functionals, then we arrive at the system

$$(\varphi_{1p_\nu}^*, \delta L\varphi - \delta f) = -\delta J_{p_\nu}. \tag{14}$$

The right-hand sides in the system of equations (14) are determined by relations (10) and are known quantities. The unknown parameters of the problem $\alpha_i = \alpha_i^{(1)} + \delta\alpha_i$ are contained in the operator δL and, possibly, in the function δf. If the operator L

and the functions f are linear with respect to the parameters α_i, then the relation

$$\delta L\varphi - \delta f = \sum_{i=1}^{n} \delta\alpha_i (A_i\varphi - \xi_i), \tag{15}$$

holds, where A_i are known operators, ξ_i are known functions. As a result, we arrive at the following system of equations for determining $\delta\alpha_i$:

$$\sum_{i=1}^{n} a_{i\nu}\delta\alpha_i = -b_\nu, \tag{16}$$

where

$$a_{i\nu} = (\varphi^*_{1p_\nu}, A_i\varphi - \xi_i), \qquad b_\nu = \delta J_{p_\nu}, \qquad \nu = 1, 2, \dots, n. \tag{17}$$

However, system (16) is still not completely determined, since its coefficients are determined with the aid of the solution of equation (1), which is not known in advance.

To determine the coefficients $a_{i\nu}, b_\nu$, one may use the method of successive approximations

$$a_{i\nu}^{(n+1)} = (\varphi^*_{1p_\nu}, A_i\varphi_n - \xi_{ni}),$$

$$b_\nu^{(n+1)} = \delta J_{p_\nu}^{(n)}, \tag{18}$$

where

$$\delta J_{p_\nu}^{(n)} = J_{p_\nu}(\varphi) - J_{p_\nu}(\varphi_n). \tag{19}$$

As a result of solving equation (1) and system (16), we obtain a sequence of solutions of the inverse problem.

$$\alpha_i^{(n+1)} = \alpha_i^{(n)} + \delta\alpha_i^{(n)}. \tag{20}$$

It is also possible to formulate another iterative process, which, as a rule, converges considerably faster:

$$a_{i\nu}^{(n+1)} = (\varphi^*_{np_\nu}, A_i\varphi_n - \xi_i),$$

$$b_\nu^{(n+1)} = \delta J_{p_\nu}^{(n)}. \tag{21}$$

The convergence of the sequence $\{\alpha_i^{(n)}\}$ to the exact solution of the inverse problem is established in each particular case.

It should be noted that the practical implementation of the solution of the system of equations (16) is a separate difficult problem, since as $n \to \infty$ the closure of the formulation of the problem leads to a problem that is ill-posed in the classical sense.

New methods for solving ill-posed problems are discussed in works ([2,3]), etc. In the present paper, the case of a linear dependence of the operator L on the parameters of the problem $\{\alpha_i\}$ has been investigated. The algorithm considered can, without substantial changes, also be applied to solving problems with a nonlinear dependence of the operator L on $\{\alpha_i\}$. In this case one may use linearization by means of Newton' s method.

The method presented above for solving inverse problems is especially effective in those cases where it is required to determine deviations of parameters from the norm, the normal value of the totality of the parameters being known. In this case, the normal values of the parameters should be taken as the predicted values.

The problem is simplified still further if, in addition, it is assumed that the deviations of the parameters of the problem from their normal values are small and that the changes in the solutions of the problem under the indicated replacement are small. In this case the inverse problem can be formulated in terms of the theory of small perturbations,

namely, the coefficients $a_{i\nu}, b_\nu$ are determined by the formulas

$$\begin{aligned} a_{i\nu} &= (\varphi_{1p_\nu}^*, A_i\varphi_1 - \xi_i), \\ b_\nu &= \delta J_{p_\nu}, \end{aligned} \tag{22}$$

where $\varphi_1, \varphi_{1p}^*$ are the solutions of equations (1) and (4) for normal or standard values of the parameters.

In conclusion, let us note that the algorithm considered in the present paper for formulating and solving inverse problems, along with information about the solution of equations (1), essentially uses the solution of the adjoint equation, which plays the role of a statistical weight in the formula of the perturbation theory for the functional (14). In this connection the function φ_p^* could be called the value of the information with respect to the functional J_p, and the adjoint equation (4) the equation for the value of the information. The equation for the value of the information may play an essential role in the planning of an experiment, when it is necessary to determine a set of functionals J_{p_ν} that make it possible, as a result, to obtain well-conditioned systems of equations of the form (16) and, consequently, to solve inverse problems effectively.

All the considerations set forth in the present paper are, of course, based on the a priori assumption of the existence and uniqueness of solutions of the inverse problems.

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Note: Figure translations are in progress. See original paper for figures.