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K. V. VALIKOV

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Abstract

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K. V. VALIKOV

CHARACTERISTIC EXPONENTS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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I. We consider the differential equation

$$\frac{dx}{dt} = Ax. \tag{1}$$

Here A is a closed linear operator with domain of definition $D(A)$, dense in the complex Banach space X, and with values in X. Suppose that there exist $M > 0$, $\theta \in (0, \pi/2)$, and a complex number z_0 such that $\sigma(A) \subset S_{\theta,z_0}$ and

$$\|R(\lambda; A)\| \leq \frac{M}{|\lambda - z_0|}, \qquad \lambda \in \overline{S}_{\theta,z_0}, \tag{2}$$

where

$$S_{\theta,z_0} = \{\lambda \mid \pi - \theta < \arg(\lambda - z_0) < \pi + \theta,\ |\lambda - z_0| > 0\},$$

$\sigma(A)$ is the spectrum of A, and $R(\lambda; A)$ is the resolvent of A. The class of such operators will be denoted by $H[X]$. It is known [1] that an operator A of class $H[X]$ generates the semigroup e^{At}, analytic for $t > 0$ and strongly continuous at $t = 0$. The solution of equation (1) satisfying the condition

$$x(t_0) = x_0, \tag{3}$$

where $x_0 \in X$, is given by the formula $x(t) = e^{A(t-t_0)}x_0$. If $x(t)$ is a function defined for $t \geq t_0$ with values in X, then, as usual, the characteristic exponent of $x(t)$ is called

$$\mu(x) = \overline{\lim_{t\to\infty}} \frac{\ln \|x(t)\|}{t}. \tag{4}$$

In the case of bounded A, the exponents of the solutions of (1) were studied by M. G. Krein [2]. He showed that $\sup\{\mu(x),\ x \text{ a solution of } (1)\}$ coincides with $\sup \operatorname{Re}\sigma(A)$ and with the lower bound of the numbers ρ for which $\|e^{At}\| \le N_\rho e^{\rho t}$, $t \ge 0$, for some $N_\rho > 0$. The following theorem generalizes M. G. Krein' s result to the case of unbounded A.

Theorem 1. If A satisfies (2), then for any solution $x(t)$ of equation (1) either $\mu(x) = -\infty$, or $\mu(x) \in \operatorname{Re}\sigma(A)$. If ξ_0 is a point of $\operatorname{Re}\sigma(A)$ isolated on the right, then there exists a sequence $x_n(t)$ of solutions of (1) for which $\mu(x_n) \le \xi_0$, $\lim_{n\to\infty} \mu(x_n) = \xi_0$. For any isolated point $\xi_0 \in \operatorname{Re}\sigma(A)$, (1) has a solution with $\mu(x) = \xi_0$. The same is true if ξ_0 is the real part of an eigenvalue of A.

The theorem can be strengthened if A is assumed to be a spectral operator. Information about such operators is given in the articles [3,4].

Theorem 2. Let A be a spectral operator with resolution of the identity $E(\delta)$ and bounded radical part N [4], and with spectrum in S_{θ,z_0}. In this case $A \in H[X]$, and if we put $E_{\xi_0} = E(\{\lambda \mid \operatorname{Re}\lambda \le \xi_0\})$, then (1) has solutions with exponent ξ_0 if and only if the fulfil-

one of the following conditions holds:

1. $E_{\xi_0} - E_{\xi_0 - 0} \ne 0$.
2. $E_{\xi_0} - E_{\xi_0 - 0} = 0$, but ξ_0 is a right limiting point of $\operatorname{Re}\sigma(A)$.

In this case the totality of initial points of solutions of (1) with exponents not exceeding ξ_0 is $E_{\xi_0}X$, and in (4) one may take the ordinary limit.

II. Put $A_0 = z_0 I - A$. If condition (2) is satisfied for the operator A_0, one can construct, as was done in (5), fractional powers A_0^α, $\alpha \ge 0$. Consider the differential equation

$$\frac{dx}{dt} = Ax + h(x,t). \tag{5}$$

Here $h(x,t)$ is defined for $t \ge 0$, $x \in D(A_0^\alpha)$ for some $\alpha \in [0,1)$, $h(0,t) \equiv 0$, and $g(x,t) = h(A_0^\alpha x, t)$ is continuous in (x,t) on $X \times [0,+\infty)$. Following (6), we introduce the following

Definition. A **generalized solution** of problem (5), (3) will mean a function $x(t)$, continuous for $t \ge t_0$, satisfying (3), and such that $x(t) \in D(A_0^\alpha)$, $t > t_0$, $A_0^\alpha x(t)$ is continuous for $t > t_0$ and integrable on every $[t_0, t_0 + T]$, and $x(t)$ satisfies, for $t > t_0$, the equation

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} h(x(\tau),\tau)\, d\tau. \tag{6}$$

Questions of existence of solutions of quasilinear equations with unbounded nonlinearities were studied by P. E. Sobolevskii in (6). Applied to equation (5), his results imply the existence of ordinary solutions under certain smoothness conditions on $h(x,t)$ and if $x_0 \in D(A_0^\beta)$ for some $\beta > \alpha$. Using methods of the perturbation theory of semigroups (1), for equation (5) one can dispense with restrictions on x_0.

Theorem 3. Let $g(x,t)$ satisfy on $[t_0, t_0+T]$ the condition

$$\|g(x_1,t) - g(x_2,t)\| \le L\|x_1 - x_2\|, \qquad x_1, x_2 \in X, \quad t \in [t_0, t_0+T].$$

In that case, for every $x_0 \in X$ there exists a unique generalized solution of problem (5), (3), defined on $[t_0, t_0+T]$. This solution will be ordinary if $g(x,t)$ satisfies the condition

$$\|g(x_1,t_1) - g(x_2,t_2)\| \le L(\|x_1 - x_2\| + |t_1 - t_2|^\beta), \qquad x_1, x_2 \in X, \quad t_1, t_2 \in [t_0, t_0+T],$$

for some $\beta \in (0,1]$.

III. The exponents of solutions of equation (5) in the finite-dimensional case have been the subject of study by many authors. The results of this section are a generalization of results obtained in (7) by D. M. Grobman. We shall assume that the condition

$$\|g(x_1,t) - g(x_2,t)\| \le \gamma(t)\|x_1 - x_2\|, \qquad x_1, x_2 \in X, \quad t \ge 0, \tag{7}$$

is satisfied, where $\gamma(t)$ is a continuous and nonnegative function for $t \ge 0$. We shall call an interval (ξ_1, ξ_2) a **gap** of $\operatorname{Re}\sigma(A)$ if $(\xi_1,\xi_2) \cap \operatorname{Re}\sigma(A)$ is empty. To a gap (ξ_1, ξ_2) there corresponds a decomposition of the space X into the direct sum of subspaces X_1, X_2 reducing A. Let A_1, A_2 be the parts of A on X_1, X_2; A_{10}, A_{20} the parts of A_0 on X_1, X_2. For any $\delta > 0$ the estimates

$$\|A_{10}^\alpha e^{A_1 t}\| \le \frac{N_1^{(\alpha)}(\delta)}{t^\alpha} e^{(\xi_1+\delta)t}, \quad t > 0; \qquad \|A_{20}^\alpha e^{-A_2 t}\| \le N_2^{(\alpha)}(\delta) e^{(-\xi_2+\delta)t}, \quad t \ge 0, \tag{8}$$

are valid for some $N_1^{(\alpha)}(\delta)$, $N_2^{(\alpha)}(\delta)$.

Theorem 4. Let (ξ_1, ξ_2) be a gap of $\operatorname{Re}\sigma(A)$, $\varepsilon \in \left(0, \dfrac{\xi_2 - \xi_1}{2}\right)$, and suppose there exists $\delta \in (0, \varepsilon]$ such that

$$N_1^{(\alpha)}(\delta)\|P_1\| \sup_{t>t_0} \int_{t_0}^{t} \frac{e^{(\delta-\varepsilon)(t-\tau)}\gamma(\tau)\,d\tau}{(t-\tau)^\alpha} + N_2^\alpha(\delta)\|P_2\| \sup_{t>t_0} \int_{t}^{\infty} e^{(\delta-\varepsilon)(\tau-t)}\gamma(\tau)\,d\tau < 1. \tag{9}$$

where P_1, P_2 are projectors onto X_1, X_2. In this case, for any $x_0 \in X_1$ equation (5) has a unique generalized solution $x(t, x_0)$ such that $P_1 x(t_0, x_0) = x_0$, $\mu(A_0^\alpha x) \leqslant \xi_1 + \varepsilon$. The transformation $\Phi : x_0 \to (t_0, x_0)$ is a homeomorphism of X_1 onto the closed set $X_{\xi_1+\varepsilon}$ of initial points of solutions of (6) for which $\mu(A_0^\alpha x) \leqslant \xi_1 + \varepsilon$. Moreover, if $x_0 \notin X_{\xi_1+\varepsilon}$ and $x(t)$ is a generalized solution of problem (5), (3), then $\mu(A_0^\alpha x) \geqslant \xi_2 - \varepsilon$.

Put

$$Q_{\delta,\varepsilon}(\gamma) = \sup_{t \geqslant t_0} \int_{t_0}^{t} \frac{e^{(\delta-\varepsilon)(t-\tau)}\gamma(\tau)\,d\tau}{(t-\tau)^{\alpha}}, \qquad R_{\delta,\varepsilon}(\gamma) = \sup_{t \geqslant t_0} \int_{t}^{\infty} e^{(\delta-\varepsilon)(\tau-t)}\gamma(\tau)\,d\tau.$$

Theorem 5. Let $\varepsilon > 0$, $\delta \in (0, \varepsilon]$ be given. There exists $\Delta = \Delta(\delta, \varepsilon) > 0$ such that, if $Q_{\delta,\varepsilon}(\gamma), R_{\delta,\varepsilon}(\gamma) < \Delta$, then for every generalized solution $x(t)$ of equation (5) either $\mu(A_0^{\alpha}x) = -\infty$, or $\mu(A_0^{\alpha}x)$ lies in the ε-neighborhood of $\operatorname{Re}\sigma(A)$, if $\operatorname{Re}\sigma(A)$ is regarded as a set on the complex sphere with the corresponding distance. If $\sigma_0 \subset \operatorname{Re}\sigma(A)$ is a bounded set, simultaneously closed and open in the relative topology of $\operatorname{Re}\sigma(A)$, then Δ can be chosen so that (5) will have solutions for which $\mu(A_0^{\alpha}x)$ lies in the ε-neighborhood of σ_0.

Corollary. Suppose that for every $\varepsilon > 0$

$$\lim_{t_0 \to +\infty} \sup_{t \geqslant t_0} \int_{t_0}^{t} \frac{e^{-\varepsilon(t-\tau)}\gamma(\tau)\,d\tau}{(t-\tau)^{\alpha}} = \lim_{t_0 \to \infty} \sup_{t \geqslant t_0} \int_{t}^{\infty} e^{-\varepsilon(t-\tau)}\gamma(\tau)\,d\tau = 0.$$

In that case, for every generalized solution of equation (5) either $\mu(A_0^{\alpha}x) = -\infty$, or $\mu(A_0^{\alpha}x) \in \operatorname{Re}\sigma(A)$. If ξ_0 is an isolated point of $\operatorname{Re}\sigma(A)$ and t_0 is sufficiently large, then there exists a solution of (5) for which

$$\mu(A_0^{\alpha}x) = \xi_0.$$

IV. Theorem 5 gives sufficient conditions under which the $\mu(A_0^{\alpha}x)$ are grouped in some manner near $\operatorname{Re}\sigma(A)$. It does not guarantee the existence of solutions (5) for which $\mu(A_0^{\alpha}x)$ lies in a neighborhood of a prescribed ξ_0, except in the case when ξ_0 is an isolated point of $\operatorname{Re}\sigma(A)$. For the case of a spectral operator A, one can obtain results concerning non-isolated points of $\operatorname{Re}\sigma(A)$.

Theorem 6. Let A be a spectral operator of scalar type (4), satisfying the conditions of Theorem 2, let $h(x,t)$ satisfy the conditions of Section III, and let $\gamma(t)$ satisfy the conditions

$$\frac{4M^2}{\cos\theta}\left[\alpha \int_{t_0}^{t} \frac{\gamma(\tau)\,d\tau}{(t-\tau)^{\alpha}} + (\operatorname{Re} z_0 - \xi)^{\alpha} \int_{t}^{\infty} \gamma(\tau)\,d\tau\right] < 1, \qquad t_0 < t \leqslant t_0 + \frac{\alpha}{\operatorname{Re} z_0 - \xi},$$

$$\frac{4M^2}{\cos\theta}\left[(\operatorname{Re} z_0 - \xi)^{\alpha} \int_{t_0}^{t-\frac{\alpha}{\operatorname{Re} z_0 - \xi}} \gamma(\tau)\,d\tau + \alpha^{\alpha} \int_{t-\frac{\alpha}{\operatorname{Re} z_0 - \xi}}^{t} \frac{\gamma(\tau)\,d\tau}{(t-\tau)^{\alpha}} + (\operatorname{Re} z_0 - \xi)^{\alpha} \int_{t}^{\infty} \gamma(\tau)\,d\tau\right] < 1, \qquad t > t_0 + \frac{}{\mathrm{Re}}$$

for all $\xi \in [\beta - T, \beta]$ for some $T > 0$, where $\beta = \sup \operatorname{Re}\sigma(A)$, and M is the upper bound of $\|E(\delta)\|$.

In that case:

1. For every generalized solution (5), either $\mu(A_0^{\alpha}x) \in [-\infty, \beta - T)$, or $\mu(A_0^{\alpha}x) \in [\beta - T, \beta] \cap \operatorname{Re}\sigma(A)$.

2. The set $\bar{X}_\xi$ of initial points of solutions of (5) for which $\mu(A_0^\alpha x) \leqslant \xi$, $\xi \in [\beta - T, \beta]$, is homeomorphic to $E_\xi X$, and if Φ_ξ is the corresponding homeomorphism, then $E_\xi \Phi_\xi x = x,\ x \in E_\xi X$.
3. For any $\xi \in [\beta - T, \beta] \cap \operatorname{Re}\sigma(A)$ there exists a sequence $x_n(t)$ of solutions of (5) such that
$$\lim_{n\to\infty} \mu(A_0^\alpha x_n) = \xi.$$

V. The results of the preceding items describe the behavior of the exponents $A_0^\alpha x$ and say nothing about the exponents of the solutions themselves. The following theorem gives a sufficient condition for equality of the exponents of x and $A_0^\alpha x$.

Theorem 7. Suppose there exist $\beta \in (\alpha, 1)$ and a function $\eta(t)$, defined and continuous for sufficiently large t, such that $\eta(t) \in (0, \eta_0]$ for some $\eta_0 > 0$, $\mu(\eta^{-1}) = 0$, and, if
$$\varphi(t) = \int_{t-\eta(t)}^{t} \frac{\gamma(\tau)\,d\tau}{(t-\tau)^\beta},$$
then $\mu(\varphi) \leqslant 0$. In that case, for every generalized solution $x(t)$ of equation (5), $\mu(x) = \mu(A_0^\lambda x)$ for all $\lambda \in [0, \beta]$. The conditions of Theorem 7 are satisfied if, for example, $\gamma(t)$ has at infinity growth no greater than polynomial.

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Moscow State University
named after M. V. Lomonosov

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