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## Abstract

## Full Text

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## ON DIFFERENTIATION AND INTEGRATION OF CONSTRUCTIVE FUNCTIONS

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All considerations in the present article are based on the constructive understanding of logical judgments ([1]). We take as fundamental the notion of a **constructive real $FR$-number** ([2]) and the notion of a **constructive function** ([3]) (see also ([4]), p. 390); henceforth objects of the indicated type will be called simply real numbers and functions. (Real numbers and functions in the sense of the corresponding definitions of classical analysis are nowhere used in essence; when these notions are used in heuristic considerations, they are called classical real numbers and classical functions.)

The notions of a continuous function, a uniformly continuous function, and also a function of bounded variation on a segment $[a, b]$ are defined in the same way as in ([4]). The notion of a differentiable function is defined in the same way as in ([5]) (see also the corrections in ([6]), p. 459); in essence this definition is equivalent to that given in ([7]). The notion of the Riemann integral is defined in the same way as in ([5]). A function is called **uniformly differentiable** on $[a, b]$ if it is differentiable on $[a, b]$, and the ratio of the increments of the function and the argument constructively converges to the derivative uniformly on $[a, b]$.

Analogous notions are introduced for constructive functions of several variables. A **constructive function of $k$ variables** is defined in the natural way as a constructive $k$-place operator from the space $E$ of real numbers into the space $E$ of real numbers ([2]), pp. 176, 208). The notions of continuous, uniformly continuous, partially differentiable functions of several variables, and also the notion of a partial derivative, are defined in the natural way, analogously to the one-dimensional case. The notion of the **total differential** is introduced analogously to how this is done in classical analysis (but with all logical relations understood constructively). A function of several variables possessing a total differential is called **differentiable**.

In what follows we consider functions of two variables (although most of the results can be generalized to the case of functions depending on any number of variables). The symbol $\Phi$ will be used to denote functions of two variables, and the symbols $f, g, f', g'$ to denote functions of one variable.

A function $\Phi$ is called **uniformly differentiable** in the rectangle* $[a, b; c, d]$ if it is differentiable in $[a, b; c, d]$ and the quantity

$$\frac{1}{\Delta\rho}\left(\frac{\partial\Phi}{\partial x}\bigg|_{(x_1,y_1)}(x_2-x_1)+\frac{\partial\Phi}{\partial y}\bigg|_{(x_1,y_1)}(y_2-y_1)\right)-(f(x_2,y_2)-f(x_1,y_1)),$$

* A system of numbers $[a,b;c,d]$ is called a **rectangle** if $a\le b,\ c\le d$; a pair of numbers $(x,y)$ **belongs** to the rectangle $[a,b;c,d]$ if $a\le x\le b,\ c\le y\le d$.

where

$$\Delta\rho=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\ne 0,$$

constructively converges to zero uniformly on $[a,b;c,d]$ as $\Delta\rho\to 0$.

A pair of functions $(f,g)$, defined on one and the same segment $[a,b]$, is called a **constructive curve** (or simply a **curve**) defined on $[a,b]$. The curve $(f,g)$ is called **uniformly continuous** (**differentiable**, **uniformly differentiable**) if both functions $f$ and $g$ are uniformly continuous (respectively differentiable, uniformly differentiable). A curve is called **rectifiable** if there exists a real number $l$ such that the length of every polygonal line inscribed in this curve does not exceed $l$, and if for every rational $\varepsilon>0$ there exists (constructively!) a polygonal line of the indicated type whose length is greater than $l-\varepsilon$. The concepts of **curvilinear integrals** of the first and second kind of a function $\Phi$ along a curve $(f,g)$ are defined analogously to the Riemann integral for functions of one variable.

**1. Differentiation and integration of functions of one variable.** A differentiable function $f$ may turn out to be nonuniformly continuous and even unbounded on a segment (see ([8])); thus, a constructively differentiable function $f$ does not always correspond to a classical function differentiable from the classical point of view. On the other hand, to a constructively nondifferentiable function there may correspond, in a certain sense, a classical function differentiable from the classical point of view. For example, a function $f$ defined on $[-1,1]$ and satisfying the conditions*

$$f(x)=x^2\sin\frac{1}{x},\qquad \text{if } x\ne 0,$$

$$f(x)=0,\qquad \text{if } x=0,$$

is constructively nondifferentiable.

**Theorem 1.** *If a function $f$ is differentiable on $[a,b]$ and its derivative is bounded above (or bounded below) on $[a,b]$, then $f$ is uniformly continuous on $[a,b]$.*

We shall say that a function $f$ is **locally uniformly continuous** on $[a, b]$ if, for every $x$ in $[a, b]$, there exists (constructively!) a rational $\varepsilon > 0$ such that $f$ is uniformly continuous on the intersection** of the segments $[a, b]$ and $[x-\varepsilon, x+\varepsilon]$. According to ([6]), Theorem 3.5, there exists a continuous function $f$ on $[0, 1]$ which is not locally uniformly continuous on $[0, 1]$.

**Theorem 2.** *Every function $f$ differentiable on $[a, b]$ is locally uniformly continuous on $[a, b]$.*

**Theorem 3.** *In order that a function $f$, defined on $[a, b]$, be uniformly differentiable on $[a, b]$, it is necessary and sufficient that it be differentiable and that its derivative be uniformly continuous on $[a, b]$.*

**Theorem 4.** *Every function $f$ uniformly differentiable on $[a, b]$ is a function of bounded variation on $[a, b]$ (moreover,*

$$\bigvee_a^b f = \int_a^b |f'(x)|\, dx.$$

**Theorem 5.** *If a function $f$ is integrable on $[a, b]$, then there exists a function $g$ such that*

$$g(x) = \int_a^x f(t)\, dt,$$

*and $g$ is differentiable on $[a, b]$, and its derivative is everywhere equal to $f(x)$.*

* A function satisfying these conditions can easily be constructed.

** By the **intersection** of segments $[a, b]$ and $[c, d]$ satisfying the conditions $b \leq c$ and $d \leq a$ is meant the segment $[\max(a, c), \min(b, d)]$.

Thus, in constructive analysis, the derivative of an indefinite integral is everywhere equal to the integrand.

**Theorem 6.** *There exists a function $f$, uniformly continuous and differentiable on $[0, 1]$, such that the derivative $f'$ is not integrable on $[0, 1]$.*

**2. Differentiation and curvilinear integration of functions of two variables.** With regard to differentiable functions of two variables one can make the same remarks as for functions of one variable (see the beginning of Section 1). Thus, for example, the function $\Phi$ such that

$$\Phi(x, y) = \frac{x^2 y}{x^2 + y^2}, \qquad \text{if } x^2 + y^2 \neq 0,$$

$$\Phi(x,y)=0, \qquad \text{if } x^2+y^2=0,$$

from the constructive point of view has no partial derivatives in $[-1,1;-1,1]$ (although from the classical point of view it has them).

**Theorem 7.** *In order that a function $\Phi$ be differentiable in a rectangle, it is necessary and sufficient that it have partial derivatives in this rectangle.*

The concept of **local uniform continuity** for functions of two variables is introduced analogously to the one-dimensional case.

**Theorem 8.** *Every function $\Phi$ differentiable in a rectangle is locally uniformly continuous in this rectangle.*

**Theorem 9.** *In order that a function $\Phi$, defined in the rectangle $[a,b;c,d]$, be uniformly differentiable in $[a,b;c,d]$, it is necessary and sufficient that it have uniformly continuous partial derivatives in $[a,b;c,d]$.*

**Theorem 10.** *If the curve $(f,g)$, defined on $[a,b]$, is rectifiable, then $f$ and $g$ are functions of bounded variation on $[a,b]$.*

**Theorem 11.** *There exists a uniformly continuous curve $(f,g)$, defined on $[0,1]$, such that $f,g$ are functions of bounded variation on $[0,1]$, but the curve $(f,g)$ is not rectifiable.*

**Theorem 12.** *Every uniformly differentiable curve is rectifiable.*

**Theorem 13.** *There exists a uniformly continuous and differentiable curve that is not rectifiable.*

**Theorem 14.** *If the function $\Phi$ is uniformly continuous in the rectangle $[a,b;c,d]$ and the curve $[f,g]$ is rectifiable and does not go outside $[a,b;c,d]$, then there exist curvilinear integrals of the first and second kind of the function $\Phi$ along the curve $(f,g)$.*

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*Note: Figure translations are in progress. See original paper for figures.*