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Abstract

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ON COMPLETELY IRREDUCIBLE REPRESENTATIONS OF THE REAL UNIMODULAR GROUP OF SECOND ORDER

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In the present paper all completely irreducible representations of the real unimodular group of second order are found up to equivalence, and an analogue of the Paley–Wiener theorem is given for spherical functions on this group. The basic methods used in the paper were first applied in ([1]).

We consider the real unimodular group G of second-order matrices

$$G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad ad - bc = 1 \tag{1}$$

$(a, b, c, d$ real), isomorphic to the group G_1 of second-order matrices (see ([2]))

$$a \in G_1; \qquad a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}; \qquad \alpha\bar{\alpha} - \beta\bar{\beta} = 1. \tag{1a}$$

Let R be the field of real numbers. We introduce notation for one-parameter subgroups of the group G:

$$U \ni u_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}; \qquad Z \ni z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}; \tag{2}$$

by K we shall denote the subgroup of matrices of the form

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}. \tag{3}$$

Every element g of the group G can be represented, and moreover uniquely, in the form $g = kz$, if $d \neq 0$.

Introduce the notation $\overline{zg} \in Z$ for the matrix z_1 such that

$$zg = kz_1. \tag{4}$$

Introduce a function on the group G:

$$\alpha(g) = |d|^{-1+i\rho}(\operatorname{sign} d)^{2j}, \tag{5}$$

where ρ is an arbitrary complex number, and j takes the value 0 or 1/2.

Define representations $T^{\rho,j}$ of the group G in the Hilbert space $L_2(Z)$ of all measurable functions $f(z)$ on the group Z satisfying the condition $\int |f(z)|^2\,dz < \infty$, where dz is the invariant measure on Z, by the formula

$$T_g^{\rho,j} f(z) = \alpha(zg) f(\overline{zg}). \tag{6}$$

In the space $L_2(-\infty, +\infty)$ the equivalent representations are

$$A_g^{\rho,j} f(z) = |bz+d|^{-1+i\rho}[\operatorname{sign}(bz+d)]^{2j} f\left(\frac{az+c}{bz+d}\right). \tag{7}$$

In the Hilbert space $\mathfrak{H} = L_2(-\pi,\pi)$ of functions $f(\psi)$ square-summable on the unit circle, we construct representations of the group G in the notation G_1 by the formula

$$T_\sigma^j(a) f(\psi) = |\bar\alpha - \beta e^{i\psi}|^{-1-2\sigma} e^{ij(\psi'-\psi)} f(\psi'), \tag{8}$$

where $\psi' = a^{-1}\psi$,

$$e^{i(a\psi)} = \frac{\bar\alpha e^{i\psi} + \bar\beta}{\beta e^{i\psi} + \alpha},$$

i.e.

$$e^{i\psi'} = \frac{\alpha e^{i\psi} - \bar\beta}{\bar\alpha - \beta e^{i\psi}};$$

$j = 0$ or $1/2$. These representations are equivalent to the representations $T_g^{\rho,j}$ considered above when j coincides and $\sigma = i\frac{\rho}{2}$.

In what follows we shall consider the representations in the form (8) (see (2)). For $\rho \in R$ these representations are unitary.

For $\rho = i(2n+2j+1)$, where n is a nonnegative integer, the representation $T^j_{i\rho/2}(a)$ is reducible. It has a finite-dimensional invariant subspace in which an irreducible representation of dimension $2n+2j+1$ is induced. In the quotient space of the representation space by this subspace there is induced a representation infinitesimally equivalent to the direct sum of two irreducible unitary representations (D^+_{m+j+1} and D^-_{m+j+1}, see (2)).

The representations $T^{\rho,j}$ and $T^{\rho',j'}$ are equivalent when $j' = j$, $\rho' = -\rho$, $\rho \neq i(2n + 2j + 1)$.

Introduce on the set $C_c^\infty(G)$ of all infinitely differentiable functions on the group G, equal to zero outside a compact set, the operations of addition and multiplication by a number in the usual way, the operation of multiplication by the formula

$$(f_1 * f_2)(g) = \int_G f_1(g_1) f_2(g_1^{-1} g)\, dg_1 \tag{9}$$

(where dg_1 is the invariant Haar measure on G), and the involution by the formula

$$f^*(g) = \overline{f(g^{-1})}. \tag{10}$$

Then the set $C_c^\infty(G)$ with the operations thus introduced becomes a ring, which we shall call the **group ring** of the group G and denote by X.

For every representation $g \to T_g$ of the group G in a Banach space $\mathfrak{H}$ one can construct a representation $f \to T_f$ of the group ring X of this group by the formula

$$T_f = \int_G f(g) T_g\, dg, \tag{11}$$

which is irreducible if the original representation of the group is irreducible.

Let $g \to T_g$ be a representation of the group G. Consider the restriction $u \to T_u$ of this representation to the subgroup U. Using the full series of irreducible unitary representations of the subgroup U, we can construct the operators

$$E^n = (2\pi)^{-1} \int_U \overline{C^n(u)}\, T_u\, du, \tag{12}$$

where

$$C^n(u_\varphi) = e^{in\varphi}. \tag{13}$$

By $\mathfrak{M}^n$ we denote the set of all vectors $\xi \in \mathfrak{H}$ satisfying the condition $E^n \xi = \xi$. The $\mathfrak{M}^n$ are closed subspaces in $\mathfrak{H}$; they are linearly independent, and $\mathfrak{H}$ is the closed direct sum of all the subspaces $\mathfrak{M}^n$.

Let $g \to T_g$ be a linear representation of the group G in a Banach space $\mathfrak{H}$. Denote by Ω the set of all finite linear combinations of vectors $\eta = T_f \xi$, where ξ ranges over all vectors from all subspaces $\mathfrak{M}^n$, and $f \in X$. The set Ω corresponding to the contragredient representation will be denoted by $\hat{\Omega}$.

Obviously, Ω is invariant with respect to the operators T_f, $f \in X$.

Suppose a representation $g \to T_g$ of the group G in the space $\mathfrak{H}$ is given; a bounded operator C in $\mathfrak{H}$ will be called **admissible** (with respect to the given representation $g \to T_g$) if it is given by a formula of the form

$$C\xi = \varphi_1(\xi)e_1 + \dots + \varphi_k(\xi)e_k, \tag{14}$$

where $\varphi_1, \dots, \varphi_k \in \hat{\Omega}$ and $e_1, \dots, e_k \in \Omega$.

We shall call the representation $g \to T_g$ **completely irreducible** if, for every admissible operator C in $\mathfrak{H}$, there exists a sequence $f_m \in X$ such that

$$(T_{f_m}\xi, \eta) \to (C\xi, \eta) \quad \text{as } m \to \infty \text{ for all } \xi \in \Omega,\ \eta \in \hat{\Omega}. \tag{15}$$

This notion coincides with the notion of irreducibility for finite-dimensional and unitary representations.

Considering the subrings with involution X_n of the ring X, defined by the condition:

$$f \in X_n, \quad \text{if } f \in X; \qquad f(u^{-1}gu_1) = C^n(u)\,\overline{C^n(u_1)}\,f(g), \tag{16}$$

we see that if the representation $g \to T_g$ of the group G in $\mathfrak{H}$ is completely irreducible, then the representation of its group ring X induces in the space $\mathfrak{M}^n \neq (0)$ a completely irreducible representation of the subring with involution X^n; moreover this representation is one-dimensional, since the rings X^n turn out to be commutative. Therefore every completely irreducible representation of the group G induces in each ring X^n a certain linear multiplicative functional. Two completely irreducible representations $g \to T_g^{(1)}$, $g \to T_g^{(2)}$ of the group G, containing one and the same representation $u \to C^n(u)$ of the group U, are equivalent if and only if, for the fixed corresponding subspaces $\mathfrak{M}^n_{(1)}$, $\mathfrak{M}^n_{(2)}$, the linear multiplicative functionals $\Lambda_1(f)$, $\Lambda_2(f)$, induced by the representations $g \to T_g^{(1)}$, $g \to T_g^{(2)}$, coincide.

To find the general form of a linear multiplicative functional in the ring X^n and to prove complete irreducibility of the representations $T^{\rho,j}$, the following analogue of the Paley–Wiener theorem for functions on the group G is needed.

For a fixed pair of integers m, n of the same parity, denote by $\mathcal{D}_{m,n}$ the totality of all functions from $C_c^\infty(G)$ satisfying the condition

$$f(u_\theta g u_\varphi) = e^{-im\theta} e^{-in\varphi} f(g). \tag{17}$$

Also denote by $U_{kl}(g, \sigma)$ the matrix elements of the representations T_σ^j in the basis $f_m = e^{i(m-j)\psi}$ (they are computed in (2) and denoted there by $u_{kl}(a)$; see (6.29), (6.23), (7.9), (7.10), (7.13), (10.27)).

Theorem 1. The following two classes of complex-valued functions $F(\sigma)$ of the complex variable σ are identical:

A. The class of all functions of σ of the form $\int_G f(g)\, u_{m/2,n/2}(g,\sigma)\, dg$, where $f(g)$ runs through all $\mathcal{D}_{m,n}$ for fixed m, n of the same parity.

B. The class of all entire functions $F(\sigma)$ of exponential type, rapidly decreasing along the line $\mathrm{Re}(\sigma) = 0$, and satisfying the conditions:

$$F(-\sigma) = \beta_{m,n} \prod_{h=1+j}^{|m|} \left(\frac{h - \frac{1}{2} - \sigma}{h - \frac{1}{2} + \sigma} \right) \prod_{h=1+j}^{|n|} \left(\frac{h - \frac{1}{2} + \sigma}{h - \frac{1}{2} - \sigma} \right) F(\sigma), \tag{18}$$

where

$$\beta_{m,n} = \begin{cases} -1, & \text{if } m \text{ and } n \text{ are odd and of different signs,} \\ 1, & \text{in all other cases;} \end{cases}$$

$$F(\sigma) = 0 \begin{cases} \text{if } m < 0,\ n \geqslant 0 \text{ for } \sigma = \dfrac{n-1}{2},\ \dfrac{n-3}{2}, \dots, -\dfrac{1}{2} + j; \\ \text{if } m \geqslant 0,\ n < 0 \text{ for } \sigma = \dfrac{1-m}{2},\ \dfrac{3-m}{2}, \dots, -\dfrac{1}{2} + j; \\ \text{if } m > 0,\ n \leqslant 0 \text{ for } \sigma = -\dfrac{n+1}{2},\ -\dfrac{n+3}{2}, \dots, -\dfrac{1}{2} + j; \\ \text{if } m \leqslant 0,\ n > 0, \text{ for } \sigma = \dfrac{m+j}{2},\ \dfrac{m+3}{2}, \dots, -\dfrac{1}{2} + j; \end{cases} \tag{19a}$$

besides,

$$F(\sigma) = 0 \begin{cases} \text{for } \sigma = -\dfrac{|m|}{2} + \dfrac{1}{2}, \dots, -\dfrac{|n|}{2} - \dfrac{1}{2} \quad \text{when } |m| > |n|; \\ \text{for } \sigma = \dfrac{|m|}{2} + \dfrac{1}{2}, \dots, \dfrac{|n|}{2} - \dfrac{1}{2} \quad \text{when } |n| > |m|. \end{cases} \tag{19b}$$

Here $j = 0$ if m, n are even, and $j = \dfrac{1}{2}$ if m, n are odd.

The theorem is a generalization of the results of ([3,4]). Using Theorem 1, the Plancherel formula found in ([5]), and following the method of ([1]), § 15, we obtain:

Theorem 2. *The completely irreducible representations of the group G are determined by a pair of numbers (ρ, j), where $\rho \neq i(2n+2j+1)$, $n = 0, \pm 1, \pm 2$.*

Here ρ is a certain complex number, while j takes the values 0 or $1/2$; moreover, the pairs (ρ, j) and $(-\rho, j)$ determine one and the same completely irreducible representation. This representation is equivalent to the representation $T^{\rho,j}$ (see (6), (7)) for these ρ and j.

The pair of numbers $(0, 1/2)$ determines two inequivalent completely irreducible unitary representations ($D^+_{1/2}$ and $D^-_{1/2}$, see [2]). To the pairs of numbers $(i(2n+2j+1), j)$ and $(-i(2n+2j+1), j)$, $n = 0, \pm 1, \pm 2, \ldots$, there correspond, when $2n+2j+1 \neq 0$, three completely irreducible representations: two unitary ones ($D^+_{\frac{|\rho|+1}{2}}$ and $D^-_{\frac{|\rho|+1}{2}}$, see [2]), and one finite-dimensional representation of dimension $|\rho| = |2n+2j+1|$. These representations are distinguished by the irreducible representations $u_\varphi \to C^n(u_\varphi)$ of the orthogonal subgroup U which they contain.

The representations listed exhaust all completely irreducible representations of the group G.

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Note: Figure translations are in progress. See original paper for figures.