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Abstract

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ON A MODEL IN THE THEORY OF SUPERCONDUCTIVITY

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In recent years a number of works have appeared in which the usual theory of superconductivity of Bardeen–Cooper–Schrieffer ([1]) and Bogolyubov ([2]) was applied to Fermi systems with attraction in the higher harmonics (see, for example, ([3], [4])). An example of such a system is liquid He^3, in which the pairs must apparently be formed in a d-state. One of the main results of applying the indicated theory is the obtaining of an anisotropic spectrum of elementary excitations.

Since it is difficult to imagine that a system with comparatively low density, such as He^3, would have an anisotropic superfluid phase, it is natural to seek a generalization of the B.C.S. model leading to an isotropic spectrum.

For this purpose we have considered a model in which the pair responsible for superconductivity is formed, in contrast to a Cooper pair, by electrons with momenta equal in absolute value but arbitrary in direction ([5]). This means selecting from the complete four-fermion Hamiltonian the terms corresponding to the carrying out of "pairings over the sphere"

$$H = \sum_{\mathbf{k},\sigma} T_k a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} - \frac{1}{V} \sum_{\mathbf{q},\mathbf{k},\mathbf{k}'} w_{\mathbf{k}\mathbf{k}'} a^+_{\mathbf{k}+} a^+_{\mathbf{q}-\mathbf{k}-} \delta_{|\mathbf{k}|\,|\mathbf{q}-\mathbf{k}|}\, a_{\mathbf{q}-\mathbf{k}'-} a_{\mathbf{k}'+} \delta_{|\mathbf{k}'|\,|\mathbf{q}-\mathbf{k}'|}. \quad (1)$$

In the interaction Hamiltonian the summation extends over vectors $\mathbf{k}$ and $\mathbf{k}'$ such that $|\mathbf{k}| \neq |\mathbf{k}'|$.

We shall not write explicitly the terms with sources; however, everywhere by averages we shall understand the quasiaverages ([6]) corresponding to averaging with the Hamiltonian

$$H' = H + \frac{\nu}{2} \sum_{\mathbf{k},\mathbf{q}} \left\{ \lambda^*_{\mathbf{k},\mathbf{q}-\mathbf{k}} a_{\mathbf{q}-\mathbf{k}-} a_{\mathbf{k}+} \delta_{|\mathbf{k}|\,|\mathbf{q}-\mathbf{k}|} + \lambda_{\mathbf{k},\mathbf{q}-\mathbf{k}} a^+_{\mathbf{k}+} a^+_{\mathbf{q}-\mathbf{k}-} \delta_{|\mathbf{k}|\,|\mathbf{q}-\mathbf{k}|} \right\}$$

and to carrying out the limiting transition $\nu \to 0$ after the transition $V \to \infty$.

It is easy to see that the operator

$$L_{\mathbf{q-k},\mathbf{k}} = \frac{1}{V}\sum_{\mathbf{k}'} w_{\mathbf{kk}'} a_{\mathbf{q-k}'-} a_{\mathbf{k}'+} \delta_{|\mathbf{k}'|\,|\mathbf{q-k}'|}$$

asymptotically as $V \to \infty$ commutes with all operators $a_{\mathbf{k}\sigma}$ and $a^+_{\mathbf{k}\sigma}$. Moreover, the presence of δ-functions in the interaction Hamiltonian restricts the summation over $\mathbf{q}$ to a surface, so that the number of vectors $\mathbf{q}$ does not exceed a quantity of order $V^{1/3}$. Both of these circumstances make it possible to hope for the existence of an asymptotically exact, as $V \to \infty$, solution for the model Hamiltonian (1).

We considered this problem by means of the method of two-time temperature Green functions and found a solution satisfying the first four equations of motion for them with accuracy $1/V^{1/3}$.

This solution can be formulated in the following way. In the Green function under consideration one must single out the operators $\beta_{\mathbf{q-k},\mathbf{k}} = a_{\mathbf{q-k}-} a_{\mathbf{k}+} \delta_{|\mathbf{k}|,|\mathbf{q-k}|}$ and their conjugates (with $\mathbf{q} \neq 0$) and, regarding them as independent of the other opera-

operators $a_{\mathbf{k}\sigma}$ and $a^+_{\mathbf{k}\sigma}$, replace averaging with the Hamiltonian H by averaging over a certain vacuum (different for the operators β and a). In practice this means satisfying the momentum conservation laws on the average, which are normal in the sense of conservation of the number of particles. The simplest averages of pairs of operators are computed with the aid of the Hamiltonian

$$H_0 = \sum_{\mathbf{k},\sigma} T_k a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + V \sum_{\mathbf{q}} \sum_{m} |\Delta^m_{\mathbf{q}}|^2 +$$

$$+ \sum_{\mathbf{q},\mathbf{k}} \delta_{|\mathbf{k}|\,|\mathbf{q-k}|} \left\{ c^*_{\mathbf{k},\mathbf{q-k}} a_{\mathbf{q-k}-} a_{\mathbf{k}+} + c_{\mathbf{k},\mathbf{q-k}} a^+_{\mathbf{k}+} a^+_{\mathbf{q-k}-} \right\}, \qquad (2)$$

where

$$w_{\mathbf{kk}'} = \sum_m \lambda^m_{\mathbf{k}} \lambda^m_{\mathbf{k}'}; \qquad \Delta^m_{\mathbf{q}} = -\frac{1}{V}\sum_{\mathbf{k}} \lambda^m_{\mathbf{k}} \langle a_{\mathbf{q-k}-} a_{\mathbf{k}+} \rangle_0; \qquad c_{\mathbf{k},\mathbf{q-k}} = \sum_m \lambda^{*m}_{\mathbf{k}} \Delta^m_{\mathbf{q}},$$

and are equal to

$$\langle a^+_{\mathbf{k}1\sigma_1} a_{\mathbf{k}2\sigma_2} \rangle_0 = \delta_{\mathbf{k}1\mathbf{k}_2} \delta_{\sigma_1\sigma_2} \frac{1}{2}\left[1 - \frac{T_k}{E_k} \operatorname{th} \frac{\beta E_k}{2}\right],$$

$$\langle a_{-\mathbf{k}1-} a_{\mathbf{k}2+} \rangle_0 = \delta_{|\mathbf{k}1|\,|\mathbf{k}_2|} \frac{c_{\mathbf{k}2,\mathbf{k}_1}}{2E_{k_1}} \operatorname{th} \frac{\beta E_{k_1}}{2}; \qquad E_k = \sqrt{T_k^2 + \sum_{\mathbf{q}} |c_{\mathbf{k},\mathbf{q-k}}|^2}.$$

Let us consider, as an example, the decoupling of a fourth-order Green function appearing in the equations of motion in the form

$$\frac{1}{V}\sum_{\mathbf{k}3,\mathbf{q}_2} w_{\mathbf{k}2\mathbf{k}_3}\left\langle\!\left\langle a^+_{\mathbf{k}3+}a^+_{\mathbf{q}2-\mathbf{k}_3-}a_{\mathbf{q}2-\mathbf{k}_2-}a^+_{\mathbf{q}1-\mathbf{k}_2-}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}a_{\mathbf{q}-\mathbf{k}_1-}a_{\mathbf{k}1+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\times$$

$$\times\delta_{|\mathbf{k}|\,|\mathbf{q}-\mathbf{k}|}\delta_{|\mathbf{k}1|\,|\mathbf{q}-\mathbf{k}_1|}\delta_{|\mathbf{k}2|\,|\mathbf{q}_1-\mathbf{k}_2|}\delta_{|\mathbf{q}-\mathbf{k}|\,|\mathbf{q}_1-(\mathbf{q}-\mathbf{k})|}\delta_{|\mathbf{k}3|\,|\mathbf{q}_2-\mathbf{k}_3|}\delta_{|\mathbf{k}2|\,|\mathbf{q}_2-\mathbf{k}_2|},\qquad |\mathbf{k}_2|\neq|\mathbf{k}_3|$$

(where $q\neq 0,\ q_1\neq 0,\ |\mathbf{k}|\neq|\mathbf{k}_1|,\ |\mathbf{k}|\neq|\mathbf{k}_2|$). In accordance with the rules formulated above, for $\mathbf{q}_2\neq 0$ we have:

$$\left\langle\!\left\langle a^+_{\mathbf{k}3+}a^+_{\mathbf{q}2-\mathbf{k}_3-}a_{\mathbf{q}2-\mathbf{k}_2-}a^+_{\mathbf{q}1-\mathbf{k}_2-}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}a_{\mathbf{q}-\mathbf{k}_1-}a_{\mathbf{k}1+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle=$$

$$=\left\langle\!\left\langle a_{\mathbf{q}-\mathbf{k}_2-}a^+_{\mathbf{q}1-\mathbf{k}_2-}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\delta_{\mathbf{q}\mathbf{q}_2}\langle a^+_{\mathbf{k}3+}a^+_{\mathbf{q}-\mathbf{k}_3-}a_{\mathbf{q}-\mathbf{k}_1-}a_{\mathbf{k}1+}\rangle=$$

$$=\delta_{\mathbf{q}\mathbf{q}_1}\delta_{\mathbf{q}\mathbf{q}_2}\left\langle\!\left\langle a_{\mathbf{k}+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a_{\mathbf{q}-\mathbf{k}_1-}a_{\mathbf{k}1+}\rangle(1-n_{\mathbf{k}2})\langle a^+_{\mathbf{k}3+}a^+_{\mathbf{q}-\mathbf{k}_3-}\rangle,$$

since $|\mathbf{k}_2|\neq|\mathbf{k}_3|$, and for $\mathbf{q}_2=0$

$$\left\langle\!\left\langle a^+_{\mathbf{k}3+}a^+_{-\mathbf{k}3-}a_{-\mathbf{k}2-}a^+_{\mathbf{q}1-\mathbf{k}_2-}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}a_{\mathbf{q}-\mathbf{k}_1-}a_{\mathbf{k}1+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle=$$

$$=\langle a_{\mathbf{q}-\mathbf{k}_1-}a_{\mathbf{k}1+}\rangle\Big\{\delta_{\mathbf{q}\mathbf{q}_1}\left\langle\!\left\langle a_{\mathbf{k}+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a_{-\mathbf{k}2-}a^+_{\mathbf{q}-\mathbf{k}_2-}\rangle\langle a^+_{\mathbf{k}3+}a^+_{-\mathbf{k}3-}\rangle+$$

$$+\delta_{\mathbf{q}\mathbf{q}_1}\left\langle\!\left\langle a_{\mathbf{k}+}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a^+_{-\mathbf{k}3-}a_{-\mathbf{k}2-}\rangle\langle a^+_{\mathbf{k}3+}a^+_{\mathbf{q}-\mathbf{k}_2-}\rangle-$$

$$-\left\langle\!\left\langle a^+_{\mathbf{q}1-\mathbf{k}_2-}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a^+_{\mathbf{k}3+}a^+_{-\mathbf{k}3-}\rangle\langle a_{-\mathbf{k}2-}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}\rangle-$$

$$-\left\langle\!\left\langle a^+_{\mathbf{q}1-\mathbf{k}_2-}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a^+_{\mathbf{k}3+}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}\rangle\langle a_{-\mathbf{k}3-}a_{-\mathbf{k}2-}\rangle-$$

$$-\left\langle\!\left\langle a^+_{-\mathbf{k}3-}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a^+_{\mathbf{k}3+}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}\rangle\langle a_{-\mathbf{k}2-}a^+_{\mathbf{q}1-\mathbf{k}_2-}\rangle+$$

$$+\left\langle\!\left\langle a^+_{-\mathbf{k}3-}\,\middle|\,a^+_{\mathbf{k}+}\right\rangle\!\right\rangle\langle a^+_{\mathbf{k}3+}a^+_{\mathbf{q}1-\mathbf{k}_2-}\rangle\langle a_{-\mathbf{k}2-}a_{\mathbf{q}1-(\mathbf{q}-\mathbf{k})+}\rangle\Big\}=0,$$

since $q\neq 0,\ q_1\neq 0,\ |\mathbf{k}|\neq|\mathbf{k}_2|,\ |\mathbf{k}_2|\neq|\mathbf{k}_3|$.

In finding an asymptotically exact solution we assume that the averages with nonconserved momentum $\langle a_{\mathbf{q}-\mathbf{k}-} a_{\mathbf{k}+} \rangle$ are quantities of order $1/V^{1/6}$ relative to $\langle a_{-\mathbf{k}-} a_{\mathbf{k}+} \rangle$. This assumption is in complete agreement with the equation for the gap (8) obtained below. In this case the condition of proportionality of the free energy to the volume is also satisfied.

Carrying out the indicated decoupling and discarding terms of order $1/V^{1/3}$, we can reduce the four equations considered to the equations

$$(E - T_k) \left\langle\!\left\langle a_{\mathbf{k}+} \,\middle|\, a^+_{\mathbf{k}+} \right\rangle\!\right\rangle_E - \sum_{\mathbf{q}} \left\langle\!\left\langle a^+_{\mathbf{q}-\mathbf{k}-} \,\middle|\, a^+_{\mathbf{k}+} \right\rangle\!\right\rangle_E c_{\mathbf{k},\mathbf{q}-\mathbf{k}} = -\frac{1}{2\pi}, \qquad (3)$$

$$(E + T_k) \left\langle\!\left\langle a^+_{q-\mathbf{k},-} \mid a^+_{k+} \right\rangle\!\right\rangle_E - \left\langle\!\left\langle a_{k+} \mid a^+_{k+} \right\rangle\!\right\rangle_E c^*_{k,\mathbf{q}-\mathbf{k}} = 0$$

and the condition

$$2T_k \langle a_{q-\mathbf{k},-} a_{k+} \rangle + (1 - 2n_k) c_{k,\mathbf{q}-\mathbf{k}} = 0, \qquad (4)$$

where

$$c_{k,\mathbf{q}-\mathbf{k}} = \frac{1}{V} \sum_{k'} w_{k\mathbf{k}'} \langle a_{q-\mathbf{k}',-} a_{k'+} \rangle \, \delta_{|\mathbf{k}'|\,|\mathbf{q}-\mathbf{k}'|}. \qquad (5)$$

The fact that, in solving the fourth equation, we have introduced no additional assumptions in comparison with the solution of the third allows one to hope that the solution obtained will be asymptotically exact for the entire chain.

Furthermore, let us note that the lowest eigenvalue of the energy for the Hamiltonian H_0 gives an upper estimate for the exact lower level of the Hamiltonian H, and the corresponding state for H_0 is isotropic. Since usually an anisotropic state lies above an isotropic one, this circumstance also speaks in favor of the truth of the solution obtained.

From (3) it is easy to obtain the spectrum of elementary excitations

$$E_k = \sqrt{T_k^2 + \Delta_k^2}, \qquad (6)$$

where

$$\Delta_k^2 = \sum_q |c_{k,\mathbf{q}-\mathbf{k}}|^2, \qquad (7)$$

and the equation for $c_{k,\mathbf{q}-\mathbf{k}}$ is obtained from (4) and (5) with the aid of the spectral representation for the function $\langle\!\langle a_{k+} \mid a^+_{k+} \rangle\!\rangle_E$ and has the form

$$c_{k,\mathbf{q}-\mathbf{k}} = -\frac{1}{V}\sum_{k'} w_{k\mathbf{k}'}\frac{c_{k',\mathbf{q}-\mathbf{k}'}}{2E_{k'}}\ \mathrm{th}\,\frac{\beta E_{k'}}{2}. \tag{8}$$

When $w_{k\mathbf{k}'} = w_{(|\mathbf{k}-\mathbf{k}'|)}$, then $c_{k,\mathbf{q}-\mathbf{k}}$ is a function only of $|\mathbf{k}|^2$ and of the angle between the vectors $\mathbf{k}$ and $\mathbf{q}$, i.e., the spectrum can in fact be isotropic. In this case the energy of the ground state coincides with the energy in the B.C.S. model in the case $l = 0$ and lies lower when $l > 0$. The transition temperature is the same in both models.

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Note: Figure translations are in progress. See original paper for figures.