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Abstract

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MATHEMATICS

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THREE TYPES OF ABSTRACT INTEGRALS

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In (^{1}) operations of formal differentiation and integration were introduced (in an abstract space X). Adding to the requirements I–III stated there certain new requirements leads to more special forms of the operations mentioned. In this way one can obtain integrals whose properties are very similar to the classical ones—the integral of an exact derivative (the Duhamel integral), the Denjoy integral, and the Lebesgue integral. This is what will be discussed.

We shall call formal differentiation **weak**, **medium**, or **strong**, and the integral generated by it, respectively, an **integral of Duhamel type**, **of Denjoy type**, or **of Lebesgue type**, if, in addition to the requirements I–III mentioned above, the following further requirements are fulfilled.

1. For **weak differentiation**:
   A. The sum of a series of derivatives uniformly convergent on $\mathcal{E}$ ($\mathcal{E} \in K$) is a derivative*.
2. For **medium differentiation**:
   B. If $f_n(x)$ ($n = 1, 2, \ldots$) are derivatives and $f_n(x) \uparrow f(x)$ as $n \to \infty$** on the set $\mathcal{E}$, and the values $F_n(\mathcal{E})$ of the corresponding primitives are bounded, then $f(x)$ is also a derivative, and for its primitive $F(E)$ the equality
   $$F(\mathcal{E}) = \lim_{n\to\infty} F_n(\mathcal{E})$$
   holds.

C. If $f(x)$ is the derivative of $F(E)$ and $\mathcal{E} \in K$, then the function

$$f_{\mathcal{E}}(x) = \begin{cases} f(x), & \text{on } \mathcal{E}, \\ 0, & \text{outside } \mathcal{E} \end{cases} \tag{1}$$

is the derivative of some function $F_{\mathcal{E}}(E)$, with $F_{\mathcal{E}}(E) = F(\mathcal{E})$ for $E \supset \mathcal{E}$.

D. $X \in K$, and there exists an additive class $\overline{K}$ of subsets of X such that: 1) $\overline{K} \supset K$; 2) the derivative is always $\overline{K}$-measurable***; 3) the characteristic function $\varphi_{\mathcal{E}}(x)$ of any set $\mathcal{E} \in \overline{K}$ is a derivative.

3. For **strong differentiation**:
   B and C—see above.
   E. The class K is additive and the derivative is always K-measurable (a strengthening of axiom D).

The corresponding integrals, besides the properties established in (^{1}), possess a number of further properties justifying their names.

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[*] In our setting derivatives are defined on the whole space X (see (^{1})). Therefore, to avoid misunderstanding, we note: the words "the function is a derivative on $\mathcal{E}$" mean that on $\mathcal{E}$ this function coincides with one of the derivatives. K is the class of subsets of X under consideration; see (^{1}).

[**] The notation $f_n(x) \uparrow f(x)$ means that $f_n(x)$ tends to $f(x)$, increasing (with respect to n); $f(x)$ is assumed finite.

[***] Here measurability is understood in the sense of (^{2}); see there also the name "additive class."

Thus, for an integral of the Duhamel type, termwise integration of a uniformly convergent series of integrable functions is possible.

For an integral of the Daniell type, axioms B and C alone already imply the following properties:

1°. If $f_n(x)$ $(n = 1, 2, ...)$ are integrable on $\mathcal{E}$, $f_n(x) \uparrow f(x)$ as $n \to \infty$, and $\int_E f_n(x)\, d\mu \leq C = \text{const}$ $(n = 1, 2, ...)$, then $f(x)$ is also integrable on $\mathcal{E}$, and

$$\int_E f(x)\, d\mu = \lim_{n\to\infty} \int_E f_n(x)\, d\mu$$

(this is a rephrasing of axiom B). Hence—the possibility of termwise integration of a series of nonnegative integrable functions (under the condition of convergence of the series of integrals or under the condition of integrability of the sum of the series).

2°. If $f(x)$ is integrable on $\mathcal{E}$, then the function (1) is integrable on every set $E \in K$, and for $E \supset \mathcal{E}$

$$\int_E f_E(x)\, d\mu = \int_E f(x)\, d\mu$$

(this is a rephrasing of axiom C). Hence—the integrability of the characteristic function of any set in K, the monotonicity of the integral of a nonnegative function and, in particular, the monotonicity of the measure μ.

3°. Finite additivity of the integral.

4°. Countable additivity of the integral of a nonnegative or bounded (at least below or above) function. Hence—countable additivity of the measure.

Let us note the following. If we want an integral of the Daniell type actually to be "similar" to the Daniell integral, then we must require that, when applied to nonnegative or to bounded functions, it lead us to something resembling the Lebesgue integral. In many respects this is indeed what happens—see 1°-4° above. However, in many respects this is not so. For example, a situation is possible in which

$$\int_E f(x)\,d\mu = 0, \qquad \mu(E) > 0,$$

but $f(x) > 0$ everywhere on E. This forces us to introduce axiom D, which we have not used so far. In this case the function

$$\bar{\mu}(E) = \int_X \varphi_E(x)\,d\mu$$

turns out to be a countably additive measure, defined on the additive class $\bar{K}$ (i.e., an ordinary measure) and is an extension of the measure μ (from the class K to the class $\bar{K}$).

Theorem 1. *Let $f(x)$ be nonnegative or bounded (at least below or above) on the set $\mathcal{E}$, $\mathcal{E} \in K$. For its integrability on $\mathcal{E}$ in the sense of an integral of the Daniell type it is necessary and sufficient that it be integrable on $\mathcal{E}$ in the sense of Lebesgue with respect to the measure $\bar{\mu}$.*

Hence, in particular, one may conclude that axiom A holds under mean differentiation. Conclusion: an integral of the Daniell type is also an integral of the Duhamel type.

The presence of many "good" properties of an integral of the Daniell type nevertheless does not allow one to assert that two derivatives of $F(E)$ can differ from one another only on a set of zero μ-measure (or $\bar{\mu}$-measure). Finally, it turns out that the measures μ and $\bar{\mu}$, generally speaking, do not determine an integral of the Daniell type uniquely.

Let us turn to the integral of Lebesgue type.

Theorem 2. *The integral of Lebesgue type coincides with the Lebesgue integral with respect to the measure μ.*

Thus: specifying the measure μ determines the integral uniquely. There are examples showing the independence of axioms B, C, E, which define strong differentiation, in other words, which define the Lebesgue integral (the situation is analogous with respect to axioms B, C, D).

Let us now consider formal differentiation satisfying the following axioms (in addition to I—III):

E.—see above.

F. If $f(x)$ is a derivative, then every function $g(x)$ differing from it on a set of μ-measure zero is also a derivative.

G. The class L of all primitives (see (^{1})) coincides with the class of all countably additive and absolutely continuous functions defined for $E \in K$.

It turns out that such differentiation coincides with strong differentiation. In other words, a formal integral for which axioms E—G are fulfilled is the μ-Lebesgue integral.

Axiom F, which at first glance may seem inessential, does not follow from the remaining axioms (I—III, E, G). Nor does the somewhat suspect property of absolute continuity of primitives required by axiom G follow from the remaining properties (see item B in (^{1})).

We shall give, in a certain sense, a constructive example of strong differentiation.

Let K be an additive class of subsets of X, and let $\mu(E)$ be a (usual) measure given on this class. Assign to the class $\overline{L}$ of primitives every function defined for $E \in K$ representable in the form

$$\Phi(E) = \sum_k t_k \mu(E \cdot E_k), \tag{2}$$

where $E_1, E_2, \ldots$ is a finite or countable system of sets from K, pairwise having no common points; $t_1, t_2, \ldots$ are arbitrary real numbers, required only so that, in the case when a series appears on the right in (2), this series be absolutely convergent.

The **derivative** of the function (2) will be called the function

$$\varphi(x) = \begin{cases} t_n, & \text{if } x \in E_n, \\ 0, & \text{if } x \in \sum_k E_k. \end{cases} \tag{3}$$

The totality of all such derivatives for all possible $\Phi(E)$ shall form the class $\overline{M}$.

It is easily verified that the mapping $\overline{L}$ onto $\overline{M}$ is a formal differentiation (i.e., axioms I—III (^{1}) are fulfilled). Consequently, putting for $E \in K$

$$\int_E \varphi(x)\, d\mu = \Phi(E),$$

we arrive at a certain formal integral. Suppose now that the sequence $\{\varphi_n(x)\}$ from $\overline{M}$ converges uniformly on X to some function $f(x)$ (which may or may not belong to $\overline{M}$). By virtue of (^{1}), for every $E \in K$ there exists a finite limit

$$\lim_{n\to\infty} \int_E \varphi_n(x)\, d\mu = F(E)$$

* There are evidently infinitely many such derivatives of $\Phi(E)$ (if only because from any of the E_n one may discard a set of measure zero; this will not change equality (2), but will change the function (3)).

(coinciding with the value of the antiderivative for $f(x)$ in the case where $f(x) \in M$).

Let us enlarge the class of antiderivatives by adjoining to it all functions obtained in this way; in doing so, we regard the derivative of $F(E)$ as the corresponding function $f(x)$. We denote this enlarged class of antiderivatives by L, and the corresponding class of derivatives by M. The mapping of L onto M, as can be verified, is a strong differentiation (the proof of this is essentially contained in (^{3})).

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REFERENCES

[1] G. P. Tolstov, DAN, **142**, No. 5 (1962).
[2] S. Saks, *Theory of the Integral*, IL, 1949.
[3] A. N. Kolmogorov, S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Moscow Univ. Press, 1960.

Note: Figure translations are in progress. See original paper for figures.