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1964

SovietRxiv

Abstract

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AN ESTIMATE FOR THE LENGTH OF A CLOSED GEODESIC IN A COMPACT RIEMANNIAN SPACE OF POSITIVE CURVATURE

(Presented by Academician I. N. Vekua on July 22, 1963)

1. In this paper we shall consider an infinitely differentiable* complete simply connected Riemannian space $R^m_{k_0}$, whose curvature at every point and in every two-dimensional direction does not exceed 1 and is not less than k_0, where k_0 is some number greater than zero.

The main result of the paper consists in the proof of the following theorem.

Theorem 1. *The length of any closed geodesic in $R^m_{k_0}$ is not less than 2π.*

This theorem for $m = 2$ was proved by A. V. Pogorelov (see ([1])). For even $m = 2n$ its proof was given by W. Klingenberg (see ([2,3])). The same author proved this theorem for $m = 2n+1$ and $k_0 > 1/4$. Thus it remained to consider the case where $m = 2n + 1$ and k_0 is an arbitrary positive number. This case will be considered in the article.

By the usual arguments, using Morse' s theorem on the distance between conjugate points in $R^m_{k_0}$ (see ([2,4])), one can show that Theorem 1 is equivalent to the following theorem:

Theorem 2. *Every arc of a geodesic of the space $R^m_{k_0}$ of length not greater than π is shortest.*

2. Let two geodesics γ_0 and $\overline{\gamma}_0$ issue from the point A_0, with endpoints at the points B_0 and $\overline{B}_0$ and lengths l_0 and $\bar{l}_0$; $l_0 < \pi$, $\bar{l}_0 < \pi$. Introduce along γ_0 and $\overline{\gamma}_0$ Fermi coordinate systems $\{x^i\}$ and $\{\overline{x}^i\}$ ([5]). Denote by A^i_k the transition matrix from the system $\{\overline{x}^i\}$ to the system $\{x^i\}$ at the point A_0. Suppose that at the points B_0 and $\overline{B}_0$ vectors h and $\overline{h}$ are given, whose coordinates are h^i and $\overline{h}^i$, and at the point A_0 a vector λ is given, whose coordinates in the system $\{x^i\}$ are λ^i, and in the system $\{\overline{x}^i\}$ are $\overline{\lambda}^i$. Let $h_1 = \lambda_1$, $\overline{h}_1 = \overline{\lambda}_1$. In the directions of the vectors λ, h, and $\overline{h}$ draw geodesics and mark on them the points $A_0(\delta)$, $B_0(\delta)$, and $\overline{B}_0(\delta)$ at distance δ from the points A_0, B_0, and $\overline{B}_0$. Join the point $A_0(\delta)$ with $B_0(\delta)$ and $\overline{B}_0(\delta)$ by geodesics $\gamma_0(\delta)$ and $\overline{\gamma}_0(\delta)$.

Denote by $\psi(\delta)$ the angle between $\gamma_0(\delta)$ and $\overline{\gamma}_0(\delta)$ at the point $A_0(\delta)$. By $\xi^i_j(x^1)$,

$\zeta^i_j(x^1)$ $(\overline{\xi}^i_j(\overline{x}^1), \overline{\zeta}^i_j(\overline{x}^1))$ denote the fundamental system of solutions of the Jacobi equations along γ_0 $(\overline{\gamma}_0)$ with initial conditions

$$\xi^i_j(0) = 0, \quad \xi^{i\prime}_j(0) = \delta^i_j, \quad \zeta^i_j(0) = \delta^i_j, \quad \zeta^{i\prime}_j(0) = 0.$$

Lemma 1.

$$\left.\frac{d\psi}{d\delta}\right|_{\delta=0} = -\left(\sum_{i=2}^{m} A^i_1 \beta^i + A^1_i \overline{\beta}^i\right) \Big/ \sin\psi(0),$$

where β^i *and* $\overline{\beta}^i$ *satisfy the system of equations:*

$$\beta^i \xi^j_i(l_0) = h^j - \lambda^i \zeta^j_i(l_0),$$

$$\overline{\beta}^i \overline{\xi}^j_i(\bar{l}_0) = \overline{h}^j - \overline{\lambda}^i \overline{\zeta}^j_i(\bar{l}_0), \qquad i, j = 2, \dots, m.$$

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* The requirement of infinite differentiability is not essential; it suffices to require twice continuous differentiability of $R^m_{k_0}$.

The proof of this lemma can be obtained by a direct calculation of the derivative $\psi'(\delta)$, using the special properties of the metric tensor in the Fermi coordinate system.

Proof of Theorem 1. We shall briefly set out the idea of the proof of Theorem 1. Suppose that Theorem 1 is false. Then there exists a closed geodesic γ of length $2l < 2\pi$. We may assume that γ has the least length among all closed geodesics. Let l_0 be such a number that $3l_0 < 2l$, $4l_0 > 2l$. Denote by Γ_ε the class of closed quadrilaterals defined by the following conditions: a quadrilateral $m \in \Gamma_\varepsilon$ if: 1) the length of each side does not exceed l_0; 2) the perimeter of m does not exceed $2l+\varepsilon$; 3) there exists a family $H_m(t)$ of closed piecewise smooth curves such that $H_m(0) = \gamma$; $H_m(1) = m$; $l(H_m(t)) < 2l + \varepsilon$. Here $\varepsilon > 0$ is an arbitrary number, and $l(H_m(t))$ is the length of the curve $H_m(t)$.

a) We shall prove that in Γ_ε there exists a quadrilateral (possibly degenerate) whose perimeter does not exceed $3l_0 < 2l$. Denote by $\psi(m)$ the smallest angle of m. Let

$$\psi_0 = \inf_{m \in \Gamma_\varepsilon} \psi(m).$$

Construct a sequence $m_k \in \Gamma_\varepsilon$ such that

$$\lim_{k \to \infty} \psi(m_k) = \psi_0.$$

Let m_0 be a limiting quadrilateral for the sequence m_k.

Two cases are possible:

I. m_0 is a degenerate quadrilateral. Then, by virtue of condition 1) in the definition of Γ_ε, the assertion of part a) is valid.

II. $\psi(m_0) = \psi_0$. We shall prove that in this case $\psi_0 = 0$. Thereby in this case also we shall prove the assertion of part a). Suppose that $\psi_0 \neq 0$. Let A_0, A_1, A_2, and A_3 ($A_4 = A_0$) be the vertices of m_0, and let ψ_0 be the angle at the vertex A_0. Denote the side $A_{iA_{i+1}}$ by γ_i, and by τ_i^+, τ_i^- the unit vectors at the points A_i, tangent to γ_i. Consider the set N of all unit vectors at the point A_0. Transport each vector of the set N parallel along m_0. By the simple connectedness of $R_{k_0}^{2n+1}$, the resulting mapping f of the set N onto itself is homotopic to the identity; and since, moreover, N is homeomorphic to the sphere S^{2n} of dimension $2n$, there exists a fixed point of the mapping f ((6), p. 604). In the language of parallel translation this means that at the point A_0 there exists a vector λ_0 which, under parallel translation along m_0, passes into itself. Let λ_i be the vector at the point A_i obtained from λ_0 by parallel translation along m_0. Define at the point A_1 (A_3) the set N_1 (N_3) of unit vectors by the conditions: $h \in N_1$ ($\bar{h} \in N_1$), if: 1) $(h, -\tau_1^-) = (\lambda_0, \tau_0^+)$ $(\bar{h}, \tau_3^+) = (\lambda_0, -\tau_0^-)$; 2) $(h, \lambda_1) \geq 1 - \varepsilon_1$ ($(\bar{h}, \lambda_3) \geq 1 - \varepsilon_1$), where (a, b) is the scalar product of the vectors a and b, and $\varepsilon_1 > 0$ is an arbitrary number.

Take two vectors $h_1 \in N_1$ and $h_3 \in N_3$. At the point A_2 find a vector h_2 such that

$$(h_1, \tau_1^+) = (h_2, -\tau_2^-), \qquad (h_3, -\tau_3^-) = (h_2, \tau_2^+).$$

In the directions of the vectors $h_0 = \lambda_0, h_1, h_2$, and h_3, draw geodesics and take on them the points $A_i(\delta)$ at distance δ from the points A_i ($\delta > 0$, if $A_i(\delta)$ lies from A_i in the direction of the vectors h_i, and $\delta < 0$ in the opposite case). Joining the points $A_i(\delta)$ and $A_{i+1}(\delta)$ by shortest arcs, we obtain a quadrilateral $m(\delta, h_i)$.

Introduce along γ_0 (γ_3) a Fermi coordinate system $\{x^i\}$ ($\{\bar{x}^i\}$). Let h^i and $\bar{h}^i$ be the coordinates of the vectors h_1 and h_3. Then, as follows from Lemma 1,

$$\left.\frac{d\psi(m(\delta, h_i))}{d\delta}\right|_{\delta=0} = -\frac{\sum A_1^i \beta^i + A_1^i \bar{\beta}^i}{\sin \psi_0}, \tag{1}$$

where β^i and $\bar{\beta}^i$ satisfy the system of equations

$$\begin{aligned} \beta^j \xi_j^i &= h^i - \lambda^j \xi_j^i, \\ \bar{\beta}^j \bar{\xi}_j^i &= \bar{h}^i - \bar{\lambda}^j \bar{\xi}_j^i, \end{aligned} \qquad i, j = 2, \dots, 2n. \tag{2}$$

If, for at least one i, $(\tau_i^+, \tau_j^-, \lambda_i) \neq 0$, then, using (1) and (2), one can prove the existence of vectors $h_1 \in N_1$ and $h_3 \in N_3$ such that $\psi'(0) \neq 0$. For definiteness let $\psi'(0) < 0$. Then, for sufficiently small δ ($\delta > 0$), in the quadrilateral $m(\delta, h_i)$

the angle at the vertex $A_0(\delta)$ is less than ψ_0. If ε_1 is taken sufficiently small, then, with the aid of Synge' s lemma ([7]), one can prove that $m(\delta, h_i) \in \Gamma_\varepsilon$.

The contradiction obtained in this case proves the assertions of part a). If, however, $(\tau_i^+, \tau_i^-, \lambda_i) = 0$ for all i, then one can prove that the plane $\tau_0^+ \wedge \tau_0^-$, determined by the vectors τ_0^+ and τ_0^-, is carried into itself under parallel displacement along m_0. But then the set of all unit vectors at the point A_0 orthogonal to the plane $\tau_0^+ \wedge \tau_0^-$ is also carried into itself under parallel displacement along m_0. Using the latter fact and the one-connectedness of $R_{k_0}^{2n+1}$, one can, as was done at the beginning of part a), prove the existence of a vector μ_0 orthogonal to the plane $\tau_0^+ \wedge \tau_0^-$ and carried into itself under parallel displacement along m_0. Thus this case is reduced to the preceding one.

b) In view of the arbitrariness of ε, from the assertion of part a) one can derive the existence of such a family $\overline{H}(t)$ of closed piecewise-smooth curves that $\overline{H}(0) = \gamma$, $\overline{H}(1) = \gamma_1$, $l(\gamma_1) \leq 3l_0 < 2l$, and $l(\overline{H}(t)) \leq 2l$ for any t. Apply to the family $\overline{H}(t)$ the Morse deformation (see, for example, ([2]), p. 71). The family $H(t)$ thereby obtained will again have the same properties as $\overline{H}(t)$, and, in addition, if for some t, $l(H(t)) = 2l$, then the curve $H(t)$ is a closed geodesic of length $2l$.

Let t_0 be the greatest value among all t for which $l(H(t)) = 2l$. For each curve $H(t)$, for $t \geq t_0$, construct a surface $F(t)$ of minimal area with boundary $H(t)$. Since $l(H(t)) < 2l$ for $t > t_0$, the theorem on the isoperimetric inequality ([9]) can be applied to them. Hence it follows that the area of the surface $F(t)$

$$S(t) < 2\pi \quad \text{for } t > t_0. \tag{3}$$

The surface $F(t_0)$ is a minimal surface; therefore its relative curvature is nonpositive, and consequently its Gaussian curvature does not exceed the maximum of the Riemannian curvature of $R_{k_0}^{2n+1}$, i.e. 1. Let us now apply the Gauss–Bonnet theorem to $F(t_0)$. It follows from it that the total curvature of $F(t_0)$ is equal to 2π, since $H(t_0)$ is a closed geodesic. But, on the other hand, the total curvature of $F(t_0)$ does not exceed $S(t_0) \cdot 1$, which, by virtue of (3), is strictly less than 2π. The contradiction obtained proves Theorem 1, and together with it Theorem 2.

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Received
12 VII 1963

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Note: Figure translations are in progress. See original paper for figures.