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Abstract

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SUFFICIENT CONDITIONS FOR ABSOLUTE STABILITY FOR ONE CLASS OF AUTOMATIC CONTROL SYSTEMS (ACS)

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Consider an ACS whose structural model consists of linear and nonlinear elements connected by a closed chain of actions: namely, if x, y, respectively, are the input and output coordinates of the nonlinear element, then y, x will be, respectively, the input and output coordinates for the linear element. The study of absolute stability and, in particular, the derivation of sufficient conditions for absolute stability is the most important stage in the analysis of such systems.

In the present paper, by the method of V. M. Popov (see, for example, ([1])), a theorem on absolute stability is proved for a class of ACS described by the properties A—for the linear part, and B and C—for the nonlinear part.

A. The linear element has a transfer coefficient of the form:

$$K(p) = \frac{1+\tau p}{T_n p^n + \cdots + T_1 p}, \qquad \tau \geqslant 0, \quad n > 1, \tag{1}$$

where $T_n p^n + \cdots + T_1$ is a Hurwitz polynomial.

B. $x(t), y(t)$ are continuous functions having a derivative almost everywhere and satisfying almost everywhere the relation

$$(\lambda \dot{y} - \dot{x})(\dot{y} - k\dot{x}) = 0, \tag{2}$$

where λ and k are real numbers satisfying the conditions $\lambda \neq 0$, $\lambda k \neq 1^*$.

C. There exists an $M < \infty$ such that $|x - y| \leqslant M$ for $xy < 0$.

From A and C, as can be shown, it follows that whatever the state $M_0(x_0, \dot{x}_0, \dots, x_0^{(n-1)}, y_0, \dot{y}_0)$ at $t = 0$, there is always such a $\bar{t}$, $0 \leqslant \bar{t} \leqslant \infty$, that $y(\bar{t}) = 0$.

An ACS described by properties A–C will be called **absolutely stable** if, for any initial state M_0, $y(t) \to 0$ as $t \to \infty$.

Theorem. *Suppose there exist numbers $\beta > \eta \geqslant 0$ such that*

$$\lambda \operatorname{Re} \Pi(i\omega) = \lambda \operatorname{Re} \{|\beta k K(i\omega)|^2 + \beta(k\lambda + 1)K(i\omega) + \lambda\eta\} \geqslant 0. \tag{3}$$

Then the system A-C is absolutely stable. For $k = 0$, it is necessary that the additional condition $\beta\lambda > 0$ be fulfilled.

1. Let us first prove the assertion: under condition (3), for any initial state M_0 of the system A-C,

$$\dot{y}(t) \to 0 \qquad \text{as } t \to \infty. \tag{4}$$

Let $\dot{y}_T(t)$ be defined as follows:

$$\dot{y}_T(t) = \begin{cases} \dot{y}(t), & 0 \leqslant t \leqslant T, \\ 0, & t < 0,\ t > T, \end{cases} \tag{5}$$

* Previously, a class of nonlinearities whose characteristics are located in a certain angle was considered (see, for example, ([1])). In the present paper a new class of nonlinearities is introduced, to which, in particular, a nonlinearity of backlash type belongs ($\lambda = 1,\ k = 0$).

and $\dot{x}_T$ for $t \leqslant T$ is the output of the linear element corresponding to the input $y(t)$ such that $\dot{y}(t) = y_T(t)$, and is zero for $t > T$. According to property A,

$$-\dot{x}_T(t) = \int_0^t G(\tau)\dot{y}_T(t-\tau)\,d\tau + y_0 G(t) + F(t, M_0), \qquad 0 \leqslant t \leqslant T. \tag{6}$$

where $G(t)$ is the impulse response of element (I), and $F(t, M_0)$ is a function depending on M_0 and $G^{(k)}(t)$ $(k = 1, 2, \dots, n)$. Obviously,

$$\int_0^\infty F^2(t, M_0)\,dt < \infty. \tag{7}$$

Consider the expression

$$\rho_T = \int_0^T \{(\beta\dot{x}_T - \eta\lambda\dot{y}_T)(\dot{y}_T - k\dot{x}_T) - (\beta k^2\lambda\dot{x}_T - k\lambda\eta\dot{y}_T)(\lambda\dot{y}_T - \dot{x}_T)\}\,dt. \tag{8}$$

Let

$$\Pi_\varepsilon(j\omega) = \beta k|K_\varepsilon(j\omega)|^2 + \beta(k\lambda + 1)K_\varepsilon(j\omega) + \lambda\eta, \tag{9}$$

where

$$K_\varepsilon(p) = \frac{1+\tau p}{T_{np}^n + \cdots + T_1 p + \varepsilon}, \qquad \varepsilon > 0, \tag{10}$$

and $\dot{Y}_T(j\omega)$, $\dot{X}_T(j\omega)$ are the Fourier transforms of the functions $\dot{y}_T(t)$ and $\dot{x}_T(t)$.

Set $y_0 = 0$. Substituting (6) into (8) and applying Parseval' s equality for the Fourier transform*, after simple transformations we obtain, for all $T > 0$ for which $y(T) = 0$:

$$\rho_T = (k\lambda - 1)\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Pi_\varepsilon(j\omega)|\dot{Y}_T(j\omega)|^2\,d\omega + 2\beta k(1-k\lambda)\int_0^T F(t, M_0)\dot{x}(t)\,dt$$
$$+ \beta(k^2\lambda^2 - 1)\int_0^T F(t, M_0)\dot{y}_T(t)\,dt + \beta k(1-k\lambda)\int_0^T F^2(t, M_0)\,dt + (k - \lambda^{-1})(\rho^2(T) + c). \tag{11}$$

Introduce into consideration the functions $\dot{y}_{T,\lambda}(t)$ and $\dot{y}_{T,k}(t)$:

$$\dot{x}_{T,\lambda} = \lambda\dot{y}_{T,\lambda}(t) = \begin{cases}\dot{y}_T(t), & \text{if } \lambda\dot{y}_T = \dot{x}_T,\\ 0, & \text{if } \lambda\dot{y}_T \neq \dot{x}_T;\end{cases} \tag{12}$$

$$k\dot{x}_{T,k} = \dot{y}_{T,k}(t) = \begin{cases}\dot{y}_T(t), & \text{if } \dot{y}_T = k\dot{x}_T,\\ 0, & \text{if } \dot{y}_T \neq k\dot{x}_T.\end{cases}$$

On the basis of (2), almost everywhere

$$\dot{y}_T^2(t) = \dot{y}_{T,k}^2(t) + \dot{y}_{T,\lambda}^2(t). \tag{13}$$

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Using here that

$$\int_0^t G(\tau)\dot{y}_T(t-\tau)\,d\tau = \lim_{\varepsilon\to 0}\int_0^t G_\varepsilon(\tau)\dot{y}_T(t-\tau)\,d\tau \quad \text{for } 0 \leqslant t \leqslant T,$$

where $G_\varepsilon(t)$ is the impulse response of element (10).

Then

$$\rho_T = \int_0^T \{(\beta\dot{x}_T - \eta\lambda\dot{y}_{T,\lambda})(\dot{y}_{T,\lambda} - k\dot{x}_{T\lambda}) -$$

$$-(\beta k\lambda^2\dot{x}_T - k\lambda\dot{y}_{T,k})(\lambda\dot{y}_{T,k} - \dot{x}_{T,k})\}\,dt = (\beta-\eta)\lambda(1-k\lambda)\int_0^T \dot{y}_T^2(t)\,dt. \quad (14)$$

From (11) and (14) we obtain that the following equality holds:

$$(k\lambda-1)\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Pi_\varepsilon(j\omega)\,|\dot{Y}_T(j\omega)|^2\,d\omega =$$
$$= (\beta-\eta)\lambda(1-k\lambda)\int_0^T \dot{y}_T^2(t)\,dt - 2\beta k(1-k\lambda)\int_0^T F(t,M_0)\dot{x}_T(t)\,dt -$$
$$-\beta(k^2\lambda^2-1)\int_0^T F(t,M_0)\dot{y}_T(t)\,dt - \beta k(1-k\lambda)\int_0^T F^2(t,M_0)\,dt + (\lambda^{-1}-k)(\rho^2(T)+c). \quad (15)$$

As is not difficult to show, for $k \neq 0$ it follows from (3) that

$$\lambda\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\Pi_\varepsilon(j\omega)\,|\dot{Y}_T(j\omega)|^2\,d\omega \geq 0,$$

and for $k = 0$ this will be true if $\beta\lambda > 0$. Then from (15) we obtain

$$-\rho^2(T) - c - \alpha_0\int_0^T \dot{y}_T(t)\,dt + \alpha_1 k\int_0^T F(t,M_0)\dot{x}_T(t)\,dt +$$
$$+\alpha_2\int_0^T F(t,M_0)\dot{y}_T(t)\,dt + \alpha_3\int_0^T F^2(t,M_0)\,dt \geq 0, \quad (16)$$

where $\alpha_0 = (\beta-\eta)\lambda^2,\quad \alpha_1 = 2\beta\lambda,\quad \alpha_2 = -\beta\lambda(k\lambda+1),\quad \alpha_3 = \beta k\lambda.$
According to (16), for $T > 0$ and such that $y(T) = 0$, the inequality

$$\int_0^T \dot{y}^2(t)\,dt \leq \frac{|\alpha_1 k|}{\alpha_0}\left|\int_0^T F(t,M_0)\dot{x}(t)\,dt\right| +$$
$$+\frac{|\alpha_2|}{\alpha_0}\left|\int_0^T F(t,M_0)\dot{y}(t)\,dt\right| + \frac{|\alpha_3|}{\alpha_0}\int_0^T F^2(t,M_0)\,dt - \rho^2(T) - c. \quad (17)$$

For the integrals on the right-hand side of (17) the following estimates hold:

$$\left|\int_0^T F(t,M_0)\dot{y}(t)\,dt\right| \leq \left[\int_0^T F^2(t,M_0)\,dt\right]^{1/2}\left[\int_0^T \dot{y}^2(t)\,dt\right]^{1/2}, \quad (18)$$

$$\left| k \int_0^T F(t, M_0) \dot{x}(t)\, dt \right| \leq s \left[\int_0^T F^2(t, M_0)\, dt \right]^{1/2} \left[\int_0^T \dot{y}^2(t)\, dt \right]^{1/2},$$

where $s = \max(1, \lambda^2 k^2)$.

Substituting (18) into (17), we obtain the inequality:

$$\left[\int_0^T \dot{y}^2\, dt \right]^{1/2} \left\{ \left[\int_0^T \dot{y}^2(t)\, dt \right]^{1/2} - \frac{|\alpha_1| s + |\alpha_2|}{\alpha_0} \left[\int_0^T F^2(t, M_0)\, dt \right]^{1/2} \right\} \leq \frac{|\alpha_3|}{\alpha_0} \int_0^T F^2(t, M_0)\, dt - -\rho^2(T) c. \quad (19)$$

from which, by virtue of (7) and properties A and C, we obtain $\lim_{t \to \infty} \dot{y}(t) = 0$ for $y_0 = 0$. On the basis of properties A, C this will also be true for arbitrary y_0. Assertion (4) is proved.

Let us show that, for a system A—C, absolute stability of the system follows from (4). We shall prove this by contradiction. Suppose $y(t)$ does not tend to zero as $t \to \infty$. Then, by virtue of properties A—C, there will be an $m > 0$ and a sequence $\{t_i\}$ $(i = 1, 2, \ldots)$ such that

$$y(t_i) = m; \qquad \dot{y}(t_i) = \mu \dot{x}(t_i),\ \mu \neq 0 \quad (20)$$

Let $\{\tilde{x}_i(t)\}$ be a sequence of functions defined for $t \geq t_i$ $(i = 1, 2, \ldots)$, satisfying the equation $T_n x^{(n)} + \cdots + T_1 \dot{x} = -m$ with initial conditions $x^{(s)}(t_i) = x^{(s)}(t_i) = x_{is}$ $(s = 1, 2, \ldots, n-1)$. Consider the sequence of intervals $\Lambda_i : [t_i, t_i + \Omega_i]$, where the quantities Ω_i are collectively bounded. Let $\Delta x_i^{(s)} = \max |\tilde{x}_i^{(s)}(t) - x^{(s)}(t)|$ for $t \in \Lambda_i$ $(s = 1, 2, \ldots, n-1)$. It can be shown that under condition (4) $\Delta x_n^{(s)} \to 0$ as $n \to \infty$ $(s = 1, 2, \ldots, n-1)$. Let $\Delta t_i = \Delta t(x_{is})$ be a time interval such that, for $t \geq t_i + \Delta t$,

$$\tilde{x}_i(t) \in \left[-\frac{3m}{2T_1} - \varepsilon,\ -\frac{m}{2T_1} + \varepsilon \right], \quad \varepsilon > 0.$$

Obviously, Δt is a continuous function of x_{is} $(s = 1, \ldots, n-1)$. Under conditions (4) and (20), the functions $x^{(s)}(t)$ and the quantities x_{is} $(s = 1, 2, \ldots, n-1;\ i = 1, 2, \ldots)$ are bounded. Therefore, by continuity of the function Δt, the sequence $\{\Delta t_i\}$ $(i = 1, 2, \ldots)$ is bounded.

Let $k \neq 0$. Put $\Omega_i = \Delta t_i$. Then there exists an N such that, for $n > N$,

$$\dot{x}(t_n + \Omega_n) \in \left[-\frac{3m}{2T_1},\ -\frac{m}{2T_1} \right],$$

whence, according to (2), we obtain that $\dot{y}(t)$ does not tend to zero as $t \to \infty$. This contradicts condition (4). Consequently, $y(t) \to 0$ as $t \to \infty$.

Let $k = 0$. According to (2), there exists a sequence of time intervals $\{\theta_i\}$ on which $\dot{y} = 0$. Let $\theta = \max_i\{\theta_i\}$. It can be shown that $\theta < \infty$. Put $\Omega_i = \Delta t_i + \theta + \varepsilon,\ \varepsilon > 0$. Then there exists $\bar{t}_i \in [t_i + \Delta t_i, t_i + \Delta t_i + \theta + \varepsilon]$ such that $\dot{x}(\bar{t}_i) = \lambda\dot{y}(\bar{t}_i)$, whence, with the aid of arguments analogous to the case $k \neq 0$, we obtain that $\dot{y}(t)$ does not tend to zero. This contradicts condition (4). Consequently, also for $k = 0$, $y(t) \to 0$ as $t \to \infty$. The theorem is proved.

Example. $n = 3$. Inequality (3) has the form:

$$\lambda^2\eta T_3^2\omega^6 + \{\lambda^2\eta(T_2^2 - 2T_1T_3) - \lambda\beta(k\lambda + 1)\tau T_3\}\omega^4 +$$

$$+\{\lambda\beta k\tau^2 + \lambda\beta(k\lambda + 1)(\tau T_1 - T_2) + \lambda^2\eta T_1^2\}\omega^2 + \beta\lambda k \leq 0.$$

The conditions of the theorem will be satisfied if the parameters of the system satisfy the conditions:

$$1)\quad \lambda > 0;\ k \geq 0,\ \lambda(T_2^2 - 2T_1T_2) - (k\lambda + 1)\tau T_3 > 0,$$
$$k\tau^2 + (k\lambda + 1)(\tau T_1 - T_2) + \lambda T_1^2 > 0;$$

$$2)\quad \lambda < 0;\ k < 0,\ k\tau^2 + (k\lambda + 1)(\tau T_1 - T_2) \leq 0,$$
$$\text{or } k\tau^2 + (k\lambda + 1)(\tau T_1 - T_2) > 0 \text{ and}$$
$$\lambda T_1^2 + k\tau^2 + (k\lambda + 1)(\tau T_1 - T_2) < 0,$$
$$-\lambda(k\lambda + 1)\tau T_1^2 T_3 - \lambda(T_2^2 - 2T_1T_3)[k\tau^2 + (k\lambda + 1)(\tau T_1 - T_2)] > 0.$$

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1. M. A. Aizerman, F. R. Gantmakher, *Absolute Stability of Control Systems*, Moscow, 1963.

Note: Figure translations are in progress. See original paper for figures.