THE SECOND BOUNDARY-VALUE PROBLEM OF THE THEORY OF ELASTICITY FOR A HOMOGENEOUS ISOTROPIC LAYER

1964

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

M. D. MARTYNENKO

THE SECOND BOUNDARY-VALUE PROBLEM OF THE THEORY OF ELASTICITY FOR A HOMOGENEOUS ISOTROPIC LAYER

(Presented by Academician A. Yu. Ishlinskii, February 14, 1964)

In this paper, by the group method, the second boundary-value problem of the theory of elasticity (the displacement problem) is solved for an elastic homogeneous isotropic layer.

1. The system of equilibrium equations of a homogeneous elastic isotropic body in displacements (the Lamé system) can be written in the following form:

$$\Delta u(x) + \tau \partial \partial' u(x) = 0, \tag{1}$$

where $\tau = (\lambda + \mu)/\mu$, λ, μ are elastic constants, Δ is the Laplace operator,

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}, \qquad \partial = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and the prime denotes transposition.

For system (1) the following problem is posed:

Determine a solution of system (1), twice continuously differentiable inside the layer $0 < x_3 < a$, and satisfying on the boundary of the layer the conditions

$$\begin{aligned} u(x_1, x_2, x_3)\big|_{x_3=0} &= f^{(1)}(x_1, x_2), \\ u(x_1, x_2, x_3)\big|_{x_3=a} &= f^{(2)}(x_1, x_2). \end{aligned} \tag{2}$$

Here $f^{(1)}$ and $f^{(2)}$ are prescribed twice continuously differentiable bounded functions (columns) satisfying the conditions:

1) $f^{(1)}(r,\varphi)$ and $f^{(2)}(r,\varphi)$ are periodic functions of the angle with period 2π (r,φ are the polar coordinates of the point (x_1,x_2)).
2) $\partial f^{(1)}/\partial\varphi$, $\partial f^{(2)}/\partial\varphi$ are finite.
3) The integrals exist

$$\int_{-\pi}^{\pi}\int_{0}^{\infty}\left|f^{(i)}(r,\varphi)\sqrt{r}\right|\,dr\,d\varphi<+\infty;$$

$$\int_{-\pi}^{\pi}\int_{0}^{\infty}\left|\frac{\partial^2 f^{(i)}(r,\varphi)}{\partial\varphi^2}\sqrt{r}\right|\,dr\,d\varphi<+\infty,\qquad(i=1,2).$$

4) $f^{(i)}(r,\varphi)$ $(i\le 2)$ in any finite interval $a\le r\le b$, for any φ, have a finite number of zeros and extrema.

2. It is known that the equations of the theory of elasticity, and consequently their solutions as well, are invariant with respect to the group of plane-parallel motions. Therefore it is natural to seek the solution among elements of minimal invariant subspaces. The elements of a minimal invariant subspace for the group of plane-parallel motions are repre-

are represented as follows:

$$u_{l,\lambda}(r,\varphi,x_3)=X_1(x_3)J_{l+1}(\lambda r)e^{i(l+1)\varphi}(e_1-ie_2)+$$
$$+X_2(x_3)J_{l-1}(\lambda r)e^{i(l-1)\varphi}(e_1+ie_2)+X_3(x_3)J_l(\lambda r)e^{il\varphi}e_3,\qquad(3)$$

where

$$e_1=\begin{pmatrix}1\\0\\1\end{pmatrix},\qquad e_2=\begin{pmatrix}0\\1\\0\end{pmatrix},\qquad e_3=\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$J_l(r)$ is the Bessel function of order l; (r,φ,x_3) are the cylindrical coordinates of the point x; $X_i(x_3)$ $(i\leqslant 3)$ are unknown functions, for the determination of which the following system of three ordinary differential equations of the second order is obtained:

$$\begin{aligned}
&X_1''-\frac{2+\tau}{2}\lambda^2X_1+\frac{\tau}{2}\lambda^2X_2-\frac{\tau}{2}\lambda X_3'=0,\\
&X_2''+\frac{\tau}{2}\lambda^2X_1-\frac{2+\tau}{2}\lambda^2X_2-\frac{\tau}{2}\lambda X_3'=0,\qquad(4)\\
&(1+\tau)X_3''+\lambda\tau X_1'-\lambda\tau X_2'-\lambda^2X_3=0.
\end{aligned}$$

This system is written in matrix form as

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1+\tau\end{pmatrix} X'' + \begin{pmatrix}0 & 0 & -\frac{\tau}{2}\lambda\\ 0 & 0 & -\frac{\tau}{2}\lambda\\ \lambda\tau & -\lambda\tau & 0\end{pmatrix} X' + \begin{pmatrix}\frac{2+\tau}{2}\lambda^2 & \frac{\tau}{2}\lambda^2 & 0\\ \frac{\tau}{2}\lambda^2 & -\frac{2+\tau}{2}\lambda^2 & 0\\ 0 & 0 & -\lambda^2\end{pmatrix} X = 0, \tag{4*}$$

where

$$X(x_3) = \begin{pmatrix}X_1(x_3)\\ X_2(x_3)\\ X_3(x_3)\end{pmatrix}.$$

Solving system (4*) by the residue method, we obtain

$$X(x_3,\lambda) = \frac{1}{\lambda} e^{\lambda x_3} \left\{ E + \frac{\tau}{4}\begin{pmatrix}3 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & 2\end{pmatrix} + \frac{\tau\lambda x_3}{4}\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ -2 & 2 & -2\end{pmatrix}\right\} C^{(1)} -$$

$$-\frac{1}{\lambda} e^{-\lambda x_3} \left\{ E + \frac{\tau}{4}\begin{pmatrix}3 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & 2\end{pmatrix} - \frac{\tau\lambda x_3}{4}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ 2 & -2 & -2\end{pmatrix}\right\} C^{(2)}, \tag{5}$$

where

$$C^{(i)} = \begin{pmatrix}C_1^{(i)}\\ C_2^{(i)}\\ C_3^{(i)}\end{pmatrix} \quad (i \leqslant 2), \qquad C^{(1)},\ C^{(2)} \text{ are constant column matrices.}$$

Then (3) is written in the following form:

$$u_{l,\lambda}(r,\varphi,x_3) = e^{il\varphi}\begin{pmatrix}e^{i\varphi} & e^{-i\varphi} & 0\\ -ie^{i\varphi} & ie^{-i\varphi} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}J_{l+1}(\lambda r) & 0 & 0\\ 0 & J_{l-1}(\lambda r) & 0\\ 0 & 0 & J_l(\lambda r)\end{pmatrix} X(x_3,\lambda). \tag{6}$$

3. Representing the solution of the required problem in the form

$$u(r,\varphi,x_3) = \sum_{l=-\infty}^{\infty}\int_0^{\infty} u_{l,\lambda}(r,\varphi,x_3)\,d\lambda, \tag{7}$$

we obtain, for determining the unknown matrices $C^{(1)}$ and $C^{(2)}$, the following system of algebraic equations:

$$A(0,\lambda)C^{(1)} + B(0,\lambda)C^{(2)} = \psi_l^{(1)}(\lambda),$$

$$A(a,\lambda)C^{(1)} + B(a,\lambda)C^{(2)} = \psi_l^{(2)}(\lambda),$$

where

$$A(x_3,\lambda) = \frac{1}{\lambda}e^{\lambda x_3}\left\{E + \frac{\tau}{4}\begin{pmatrix}3 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & 2\end{pmatrix} + \frac{\tau\lambda x_3}{4}\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ -2 & 2 & -2\end{pmatrix}\right\},$$

$$B(x_3,\lambda) = -\frac{1}{\lambda}e^{-\lambda x_3}\left\{E + \frac{\tau}{4}\begin{pmatrix}3 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & 2\end{pmatrix} + \frac{\tau\lambda x_3}{4}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ 2 & -2 & -2\end{pmatrix}\right\},$$

$$\psi_l^{(1)} = \begin{pmatrix}\dfrac{\tilde f_{1,l+1}^{(1)}(\lambda) + i\tilde f_{2,l+1}^{(1)}(\lambda)}{2}\\[2ex] \dfrac{\tilde f_{1,l-1}^{(1)}(\lambda) - i\tilde f_{2,l-1}^{(1)}(\lambda)}{2}\\[2ex] \dfrac{\tilde f_{3,l}^{(1)}(\lambda)}{2}\end{pmatrix},$$

$$\psi_l^{(2)} = \begin{pmatrix}\dfrac{\tilde f_{1,l+1}^{(2)}(\lambda) + i\tilde f_{2,l+1}^{(2)}(\lambda)}{2}\\[2ex] \dfrac{\tilde f_{1,l-1}^{(2)}(\lambda) - i\tilde f_{2,l-1}^{(2)}(\lambda)}{2}\\[2ex] \dfrac{\tilde f_{3,l}^{(2)}(\lambda)}{2}\end{pmatrix},$$

$$\tilde f_l^{(i)}(\lambda) = \begin{pmatrix}\tilde f_{1,l}^{(i)}\\ \tilde f_{2,l}^{(i)}\\ \tilde f_{3,l}^{(i)}\end{pmatrix}_\lambda = \int_0^\infty \left\{\int_0^{2\pi} e^{-il\theta} f^{(i)}(r,\theta)\,d\theta\right\} J_l(\lambda r)\,r\,dr \qquad (i = 1,2).$$

Having determined the constant matrices $C^{(1)}, C^{(2)}$ (columns) and substituting them into (7), we obtain

$$u(r,\varphi,x_3)=\sum_{l=-\infty}^{\infty}\begin{pmatrix} e^{i\varphi} & e^{-i\varphi} & 0\\ -ie^{i\varphi} & ie^{-i\varphi} & 0\\ 0 & 0 & 1\end{pmatrix} e^{il\varphi}\int_0^{\infty}\begin{pmatrix} J_{l+1}(\lambda r) & 0 & 0\\ 0 & J_{l-1}(\lambda r) & 0\\ 0 & 0 & J_l(\lambda r)\end{pmatrix}\begin{pmatrix} X_1(x_3,\lambda)\\ X_2(x_3,\lambda)\\ X_3(x_3,\lambda)\end{pmatrix} d\lambda.$$

The resulting series converges under the restrictions imposed on the functions $f^{(1)}(r,\varphi)$, $f^{(2)}(r,\varphi)$.

Lviv State University
named after Ivan Franko

Received
10 I 1964

Note: Figure translations are in progress. See original paper for figures.