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Abstract

Full Text

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ON THE COHOMOLOGY OF THE STEENROD ALGEBRA

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The Adams spectral sequence modulo p of the zero-dimensional sphere has as its second term the algebra $H(A)$ of cohomology of the Steenrod algebra A modulo p, and therefore, for determining the p-components $G_i(p)$ of the stable homotopy groups of spheres $G_i = \pi_{n+i}(S^n)$, $n > i+1$, knowledge of the algebra $H(A)$ becomes very important. In this direction a number of valuable results ($^{1-3}$) were obtained with the aid of the series of spectral sequences constructed by Adams. However, already the first of these sequences, because of serious technical difficulties, has still not been completely computed.

Theorem 1 of the present note asserts the existence of a spectral sequence converging to the algebra $H(A)$, whose first term E_1 is a polynomial algebra. The computation of the differentials of this sequence makes essential use of spectral cohomology operations Sq_r^i and Massey products of various orders in the terms E_r. We apply this spectral sequence to the computation of the algebra $H(A)$, $p = 2$, in dimensions $t - s < 22$.

With the help of the known properties of the J-homomorphism, these computations easily give the orders of the groups $G_{20}(2) = z_8$ and $G_{21}(2) = z_2 + z_2 + z_2$. A complete description of the group $G_{21}(2)$ follows from Toda' s results on the groups $G_i(2)$, $i < 20$, and the properties of secondary composition (^{4}).

Let A be a connected locally finite graded associative and coassociative Hopf algebra over the field z_2 with commutative diagonal, (Γ, δ) the cobar construction over A, and A^* the algebra dual to the algebra A. According to Borel' s theorem, in the algebra A^* one can choose homogeneous elements ξ_k of height 2^{c_k} and dimension t_k, $k \geqslant 1$, such that the monomials

$$\xi_1^{n_1}\xi_2^{n_2}\cdots\xi_k^{n_k}, \quad 0 \leqslant n_j < 2^{c_j}, \quad 1 \leqslant j \leqslant k,$$

form its z_2-basis. The tensor degrees of the Milnor-Moore filtration F^p of the algebra A^*, $F^0 = A^*$, $F^{p+1} = A^*F^p$, $p \geqslant 0$, determine in Γ a homogeneous, with respect to the gradings s and t, decreasing filtration $F^p\Gamma$, $p \geqslant 0$, for which $[\alpha_1|\alpha_2|\cdots|\alpha_s] \in F^p\Gamma$ if $\alpha_i \in F^{p_i}$, $p_1 + \cdots + p_s \geqslant p$. In this situation the relations

$$F^0\Gamma = \Gamma, \qquad \bigcap F^p\Gamma = 0, \qquad \delta(F^p\Gamma) \subset F^p\Gamma, \qquad F^p\Gamma \cdot F^q\Gamma \subset F^{p+q}\Gamma,$$

hold, by virtue of which there arises a spectral sequence $\{E_r, d_r\}$ of triply graded algebras

$$E_r = \sum E_r^{p,s,t}, \qquad d_r(E_r^{p,s,t}) \subset E_r^{p+r,s+1,t}.$$

The associated linear space $E = \sum F^p/F^{p+1}$ is naturally defined as a Hopf algebra, and, as is easy to see, the cobar construction over the dual algebra E^* coincides with the differential algebra (E_0, d_0). Consequently, the algebra E_1 is isomorphic to the cohomology algebra of E^* and can be computed by the methods of (^{1}). Denote by $h_{k,i}$ the images in the algebra E_1 of the elements $[\xi_k^{2^i}]$ from Γ. Then there holds

Theorem 1. The spectral sequence $\{E_r, d_r\}$ converges to the algebra $H(A)$. Its first term E_1 is the polynomial algebra with generators $h_{k,i}$ of dimension $(2^i, 1, 2^i k)$, $0 \le i < c_k,\ k \ge 1$.

Following the ideas of [1], consider the A-linear mappings $\Phi_i : B \to B \otimes B$ of degree $i,\ i \ge 0$, defined inductively by the relations

$$\Phi_0(1) = 1 \otimes 1, \quad \Phi_{i+1}(1) = 0, \quad \Phi_i s = (s \otimes s)p\Phi_{i-1} + T\Phi_i, \quad i \ge 0,$$

where s is the contracting homotopy of the B-construction B over the algebra A, $T = s \otimes 1 + \varepsilon \otimes s,\ \Phi_{-1} = 0,\ p(b_1 \otimes b_2) = b_2 \otimes b_1,\ b_1, b_2 \in B$.

Lemma 1. $d\Phi_i + \Phi_i d = (1 + p)\Phi_{i-1},\ i \ge 0$.

It follows from this lemma that the mappings $D_i : \Gamma \otimes \Gamma \to \Gamma$, dual to the mappings $\bar{\Phi}_i : \bar{B} \to \bar{B} \otimes \bar{B}$, $\mathrm{Im}(\Phi_i + \bar{\Phi}_i) \subset \bar{B} \otimes B + B \otimes \bar{B}$, where $\bar{B}$ is the reduced B-construction, satisfy the Pontryagin–Steenrod formula

$$\delta D_i + D_i \delta = D_{i-1}(1 + p), \quad i \ge 0, \quad D_{-1} = 0.$$

Consider the mappings $\mathrm{sq}^i : \Gamma \to \Gamma,\ i \ge 0$, defined by the relation

$$\mathrm{sq}^i(x) = D_{s-i}(x \otimes x) + D_{s+1-i}(x \otimes \delta x), \quad x \in \Gamma^s,$$

Lemma 2. $\delta\, \mathrm{sq}^i = \mathrm{sq}^i \delta,\ \mathrm{sq}^i(F^p\Gamma) \subset F^{2p}\Gamma,\ i \ge 0,\ p \ge 0.$

It is easy to see that the mappings sq^i define spectral cohomology operations

$$\mathrm{Sq}_r^i : E_r^{p,s,t} \to E_{2r}^{2p,s+i,2t}, \quad 1 \le r \le \infty,$$

as well as certain cohomology operations in the algebra $H(A)$, inducing in the term E_∞ the operations Sq_∞^i and coinciding with the already known operations Sq^i (see [3]).

Theorem 2. The operations Sq_r^i have the following properties:

1) $\mathrm{Sq}_r^i(x) = 0,\ i > s,\ \mathrm{Sq}_r^s(x) = \chi_{2r}^r(x^2),\ x \in E_r^s$;

2) $\mathrm{Sq}_r^i d_r = d_{2r}\mathrm{Sq}_r^i,\ \mathrm{Sq}_{r+1}^i \chi_{r+1}^r = \chi_{2r+2}^{2r}\mathrm{Sq}_r^i$;

3) $\mathrm{Sq}_r^k(xy) = \sum \mathrm{Sq}_r^i(x)\mathrm{Sq}_r^{k-i}(y)$;

4) $\mathrm{Sq}_r^0 = h_*^{(r)}$,

where the mapping $h_*^{(r)}$ is induced by the homomorphism $h : \Gamma \to \Gamma,\ h([\alpha_1|\dots|\alpha_s]) = [\alpha_1^2|\dots|\alpha_s^2]$.

Let R be an associative differential ring of characteristic 2, and let $\alpha_i,\ 0 \le i \le n$, be its cohomology classes. Suppose that there exist elements $m_{ij} \in R,\ 0 \le i \le j \le n,\ j - i \ne n$, such that m_{ii} is a cycle of class α_i and $\delta m_{ij} = \sum m_{ik} m_{k+1,j}$. Then the element $m = \sum m_{0k} m_{k+1,n}$ is a cycle of the ring R, and the set $\langle \alpha_0, \dots, \alpha_n \rangle$ of cohomology classes of all such cycles is called the Massey product of order n. We shall say that the product $\langle \alpha_0, \dots, \alpha_n \rangle$ is strictly defined if every Massey product $\langle \alpha_i, \dots, \alpha_j \rangle,\ 0 \le i < j \le n,\ j - i \ne n$, is defined and consists of the single zero.

The associative multiplications present in the differential algebras Γ and $E_{r-1},\ r \ge 1$, allow one to consider, in their cohomology algebras $H(A)$ and E_r, Massey products of various orders.

Theorem 3. Let the elements $a_i \in E_r^{p_i,s_i,t_i},\ 0 \le i \le n$, be contained in $E_{r,\infty}$, where $E_{r,\infty}$ is the subalgebra of cycles of all differentials, and let $a_i \in H(A)$ be such representatives of the elements $\chi_\infty^r(a_i)$, respectively, that the Massey product $L = \langle a_0, \dots, a_n \rangle$ is strictly defined. If the subspaces

$$E_{r+k}^{p_i+\cdots+p_j-r(j-i)-k,\ s_i+\cdots+s_j-(j-i),\ t_i+\cdots+t_j}, \quad 0 \le i < j \le n,\ j - i \ne n,\ k \ge 0,$$

are contained in $E_{r+k,\infty}$, then the Massey product $K = \langle \alpha_0, \dots, \alpha_n \rangle$ is defined and has nonempty intersection with $E_{r,\infty}$. Moreover, in $K \cap E_{r,\infty}$ there exists an element whose image under the homomorphism χ_∞^r has a representative in L.

Let us take as the algebra A the Steenrod algebra modulo 2. Applying Theorems 1, 2, and 3, one can prove the following theorem.

Theorem 4. In dimensions $t - s < 22$ the z_2-basis of the algebra $H(A)$ consists of the monomials

$$h_1 l_2^2,\ h_1\omega_{20},\ h_3\beta_{14},\ h_0^k\omega_{20},\ \alpha_{19},\ h_0^k\omega_{19},\ h_4 h_2 h_0^k,$$

$$h_0^\varepsilon\beta_{18},\ h_1^\varepsilon\omega_{17},\ h_4 h_1^k,\ \gamma_{17},\ h_2 h_0^k\beta_{14},\ h_1^\varepsilon\alpha_{16},\ h_1^{1+\varepsilon}\beta_{14},$$

$$h_4 h_0^l,\ h_3^2 h_0^\varepsilon,\ h_0^k\beta_{14},\ h_0^k\omega_{11},\ h_1^\varepsilon\omega_9,\ h_3 h_1^k,\ h_1^\varepsilon\alpha_8,$$

$$h_3h_0^{1+k},\ h_2^{\varepsilon},\ h_2h_0^{k},\ h_1^2,\ h_1,\ h_0^{n},$$

where $\varepsilon = 0, 1;\ 0 \le k \le 2;\ 1 \le l \le 7;\ n \ge 0$, from the elements h_i and certain elements

$$\alpha_8,\ \omega_9,\ \omega_{11},\ \beta_{14},\ \alpha_{16},\ \gamma_{17},\ \omega_{17},\ \beta_{18},\ \omega_{19},\ \alpha_{19},\ \omega_{20}$$

of dimensions $(s,t) = (3,11), (5,14), (5,16), (4,18), (7,23), (4,21), (9,26), (4,22), (9,28), (3,22), (4,24)$, respectively, satisfying the relations

$$h_1^2\omega_{17} = h_0^2\omega_{19}, \qquad h_1\gamma_{17} = h_0\beta_{18}, \qquad h_2\omega_{11} = h_0^2\beta_{14},$$

$$h_1^2\omega_8 = h_0^2\omega_{11}, \qquad h_3\omega_9 = h_1^2\beta_{14}, \qquad \alpha_8^2 = h_1^2\beta_{14},$$

$$\alpha_8\omega_9 = h_1\alpha_{16}, \qquad \omega_9^2 = h_1\omega_{17}, \qquad h_2^2\beta_{14} = 0.$$

In addition, $h_2\alpha_{19} \ne 0,\ H^{3,25}(A) = 0$, and

$$\alpha_8 \in \langle h_2^2, h_0, h_1\rangle, \qquad \omega_9 \in \langle h_3h_0^3, h_0, h_1\rangle,$$

$$\omega_{11} \in \langle h_2, h_0^3, h_3, h_0\rangle, \qquad \beta_{14} \in \langle \alpha_8, h_0, h_1, h_2\rangle,$$

$$\alpha_{16} \in \langle h_2^2, h_0, \omega_9\rangle, \qquad \alpha_{19} \in \langle h_3, h_3h_1, h_2\rangle,$$

$$\omega_{17} \in \langle h_3h_0^3, h_0, \omega_9\rangle, \qquad \omega_{19} \in \langle h_3h_0, h_0^3, \omega_{11}\rangle,$$

$$\beta_{18} \in \langle h_3^2, h_0^2, h_2^2\rangle.$$

Identifying the algebra $H(A)$ with the second term E_2 of the Adams spectral sequence modulo 2 of the zero-dimensional sphere, we obtain

Corollary 1. The elements $\alpha_{16}, \omega_{17}, \omega_{19}, h_2\alpha_{19}$ of the algebra E_2, as well as its elements of dimensions $t - s < 15$ and $t - s = 21$, are cycles of all differentials. The images under the homomorphism χ_∞^2 of the elements $h_4h_2^2,\ h_1\omega_{20},\ h_3\beta_{14}$, and $h_2\alpha_{19}$ are nonzero.

Since the basis of the algebra E_2 in dimensions $t-s = 21$ consists of the elements $h_4h_2^2,\ h_1\omega_{20}$, and $h_3\beta_{14}$, it follows that the group $G_{21}(2)$ has order 8.

Let ω_k be a generator of the group I_{4k-1}, where I_i is the image of the I-homomorphism in the group G_i. According to the known results of K-theory, its order is divisible by 8. From the relation

$$I(\alpha \circ \beta) = I(\alpha) \circ E^n \beta, \qquad \alpha \in \pi_i(SO(n)), \quad \beta \in \pi_{i+m}(S^i)$$

(see (4)) it follows that $I_l \circ G_m \subset I_{l+m}$, $m < i_0$, and therefore $I_{19} \circ G_3 \subset I_{22} = 0$.

Next, $\chi^2_\infty(h_2\alpha_{19}) \neq 0$, and the basis of the algebra E_2 in dimensions $t - s = 19$ consists of the elements $\alpha_{19}, h_0^k \omega_{19}, 0 \leq k \leq 2$. Consequently, ω_5 is a representative of the element $\chi^2_\infty(\omega_{19})$, and the element ω_{20} is a cycle of all differentials. Denote by $\eta, \nu, \sigma, \chi, \omega$, and α the representatives in $G(2)$ of the images under the homomorphism χ^2_∞ of the elements $h_1, h_2, h_3, \beta_{14}, \omega_{20}$, and $h_4 h_2^2$.

Corollary 2. The group $G_{20}(2)$ is a cyclic group of order 8 with generator ω. The group $G_{21}(2)$ is generated by the elements $\eta\omega$, $\sigma\chi$, and α, the first two of which have order 2.

Comparing Theorem 4 with Toda' s results on the groups $G_i(2)$, $i < 20$, we prove the following theorem.

Theorem 5. The differentials d_r in dimensions $t - s < 22$ are completely determined by the relations

$$d_2(h_4) = h_3^2 h_0, \qquad d_2(\gamma_{17}) = h_1^2 \beta_{14}, \qquad d_2(\beta_{18} h_0^\varepsilon) = h_2 h_0^{1+\varepsilon} \beta_{14}, \qquad \varepsilon = 0, 1,$$

$$d_3(\chi_3^2(h_4 h_0)) = \chi_3^2(h_0 \beta_{14}).$$

Furthermore, the secondary composition $\langle 2\sigma, \sigma, \nu \rangle$ contains a representative ν^* of the element $\chi^2_\infty(h_4 h_2)$.

Obviously, one may assume that $\alpha = \nu\nu^*$. Then

$$2\alpha = 2\nu \circ \nu^* \langle 2\sigma, \sigma, \nu \rangle 2\nu = 2\sigma \langle \sigma, \nu, 2\nu \rangle = 0,$$

since $2G_{14}(2) = 0$. Thus it has been proved

Corollary 3. $G_{21}(2) = z_2 + z_2 + z_2$.

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Note: Figure translations are in progress. See original paper for figures.