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Abstract

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GENERAL METHODS OF "MULTIPLIER ENLARGEMENT" AND APPROXIMATION OF UNBOUNDED CONTINUOUS FUNCTIONS BY CERTAIN CONCRETE POLYNOMIAL OPERATORS

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In the present paper we develop a method that we have called the method of **multiplier enlargement** and apply it to the problem of approximation by polynomials of unbounded continuous functions. For these purposes a class of the most effective polynomial operators is introduced, and a certain generalization of the interpolation polynomials of Fejér—Hermite is given. Some, apparently new, asymptotic formulas for the rate of approximation are also established.

1. Our general method can be formulated most conveniently for the case of linear positive operators. Let E denote a Banach space, and let $f(t)$ be a functional defined on E. Let $L_n(f(t), x)$ $(n = 1, 2, ...)$ be a sequence of linear positive operators transforming $f(t)$ into functionals of x $(x \in E)$. We introduce the following **restricting function** $\Omega(\|x\|)$:

$$\Omega(\|x\|) \geqslant 1, \qquad \Omega(\|x\|) \uparrow \infty \quad (\|x\| \uparrow \infty).$$

We define $C(\Omega) = \{f\}$ as the class of continuous functionals $f(x)$ $(x \in E)$ satisfying the following conditions: 1) $|f(x)|$ is bounded in each bounded closed sphere of E; 2) $f(x)$ satisfies the condition $f(x) = O(\Omega(\|x\|))$ $(\|x\| \to \infty)$.

Theorem 1. Let $\{\alpha_n\}$ be an increasing sequence of positive numbers such that $\alpha_n \to \infty$ as $n \to \infty$. Suppose that the conditions (the so-called normalization condition and singularity condition, respectively)

$$1) \quad L_n[1, \alpha_n^{-1} x] \to 1 \qquad (n \to \infty),$$

$$2) \quad L_n[\|\alpha_n t - x\|^2 \Omega(\|\alpha_n t\|); \alpha_n^{-1} x] \to 0 \qquad (n \to \infty)$$

are fulfilled uniformly in each closed sphere of E. Then for every $f \in C[\Omega]$ and every $x \in E$ we obtain:

$$\lim_{n\to\infty} L_n[f(\alpha_n t); \alpha_n^{-1} x] = f(x). \tag{1}$$

Moreover, if E is finite-dimensional, the limiting relation (1) is fulfilled uniformly in every bounded subset of E.

The proof of this theorem essentially reproduces the proof of Theorem 1 of paper ([1]), with the corresponding generalizations.

This theorem may be used as a general principle for checking the convergence of various generalizations of certain well-known polynomial operators defined on the interval $(-\infty, +\infty)$. In particular, some results of X. L. Zhudovskii, Szői, Radecki, and Mamedov ([1–4]) may be regarded as consequences of this theorem with specially chosen α_n, $\Omega(\cdot)$, etc. New applications of this theorem are given below.

2. For the Euclidean space E_k we put $t = (t_1, \dots, t_k)$, $\|t\| = \sqrt{t_1^2 + \dots + t_k^2}$; $\alpha_n t = (\alpha_n t_1, \dots, \alpha_n t_k)$, and $dt = dt_1 \cdots dt_k$ (the element

volume). The symbol $\nu = (\nu_1, \dots, \nu_k)$ always denotes a lattice point with integer components.

Let $S(\|t\| \leqslant 1)$ and $Q(|t_i| \leqslant 1;\ i = 1, \dots, k)$ be, respectively, the spherical and cubic domains in E_k.

We construct the following polynomial operators for continuous functions $f(x) = f(x_1, \dots, x_k)$ $(x \in E_k)$ with increasing multipliers $\alpha_n \uparrow \infty$ $(n \uparrow \infty)$:

$$L^1[f(\alpha_n t); \alpha_n^{-1} x] = \left(\frac{n}{\pi}\right)^{k/2} \int_S f(\alpha_n t) \left[1 - \|t - \alpha_n^{-1} x\|^2\right]^n dt;$$

$$L_n^2[f(\alpha_n t); \alpha_n^{-1} x] = \left(\frac{1}{n\pi}\right)^{k/2} \sum_{\|\nu\| \leqslant n} f\left(\frac{\alpha_n}{n}\nu\right) \left[1 - \left\|\frac{1}{n}\nu - \frac{1}{\alpha_n} x\right\|^2\right]^n;$$

$$L_n^3[f(\alpha_n t); \alpha_n^{-1} x] = \left(\frac{n}{k\pi}\right)^{k/2} \int_Q f(\alpha_n t) \left[1 - \frac{1}{k}\|t - \alpha_n^{-1} x\|^2\right]^n dt.$$

These operators may be called a **generalization of the second type of Landau polynomial operators** ([4,5]).

Theorem 2. Let $\alpha_n = n^\theta \left(0 < \theta < \frac{1}{m+2},\ m \text{ a positive integer}\right)$. Then for every continuous function $f(x) = O\left(e^{\|x\|^m}\right)$ $(\|x\| \to \infty)$ we have $(i = 1, 2, 3)$

$$\lim_{n\to\infty} L_n^{(i)}[f(n^\theta t); n^{-\theta} x] = f(x) \quad (x \in E_k). \tag{2}$$

Theorem 3. For every continuous function $f(x) = O\left(e^{\|x\|^{\|x\|}}\right)$ $(\|x\| \to \infty)$ we have

$$\lim_{n\to\infty} \left(\frac{n}{\pi}\right)^{k/2} \int_S f(t \log\log n) \left[1 - \left(t - \frac{x}{\log\log n}\right)^2\right]^n dt = f(x). \tag{3}$$

Moreover, the limiting relation (3) (as well as (2)) holds uniformly on every bounded subset of E.

In fact, Theorems 2 and 3 are particular consequences of Theorem 1, where $E = E_k$ and $\Omega(x) = e^{\|x\|^m}$ for $\alpha_n = n^\theta$; $\Omega(x) = e^{\|x\|^{\|x\|}}$ for $\alpha = \log\log n$. A more detailed verification is omitted here.

3. For every function $f(x) \in C^2$ $(x \in E_k)$ we denote by $\Delta_k f$ the Laplace operator

$$\Delta_k f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_k^2}.$$

A more delicate investigation leads to the following theorem.

Theorem 4. For every function $f(x) \in C^2$, for which $f(x) = O\left(e^{\|x\|^{\|x\|}}\right)$ $(x \to \infty)$, we have, as $n \to \infty$,

$$L_n^1\left[f(t \log\log n); \frac{x}{\log\log n}\right] - f(x) \sim \left(\frac{\log\log n}{4n}\right)^2 \Delta_k f; \tag{4}$$

$$L_n^3\left[f(t \log\log n); \frac{x}{\log\log n}\right] - f(x) \sim \frac{k(\log\log n)^2}{4n} \Delta_k f. \tag{5}$$

Remark 1. Results parallel to Theorems 2, 3, and 4 can be obtained analogously for two types of generalizations of Bernstein polynomials to spaces whose dimension is greater than one (for the definitions of these polynomials, see, for example, ([6])).

Remark 2. Our asymptotic formulas (4) and (5) are similar to Voronovskaya' s formula for Bernstein polynomials on the interval $[0, 1]$, although the orders in n here differ slightly.

Remark 3. Theorems 3 and 4 suggest that still weaker restrictions can be imposed on the growth of $f(x)$ (as $\|x\| \to \infty$) if the sequence of multiplicators increases more slowly than $\{\log\log n\}$. We can confirm this supposition, but the presentations become very complicated and delicate. Some results of this kind can also be obtained for one generalization of the Fejér–Hermite polynomials (see § 4).

4. In this paragraph we consider functions defined on $E_1 = (-\infty, +\infty)$. Then for every function $f(x)$, defined on $[-1, 1]$, we can write the corresponding Fejér interpolation polynomial of degree $2n-1$ in the form:

$$F_n[f(t); x] = \frac{1}{n^2} \sum_{\nu=1}^{n} \left(1 - a_\nu^{(n)} x\right) \left[\frac{T_n(x)}{x - a_\nu^{(n)}}\right]^2 f\left(a_\nu^{(n)}\right),$$

where $a_\nu^{(n)}$ are the zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. Introduce increasing multiplicators $0 < \lambda_1 < \cdots < \lambda_n \to \infty$. We can construct a modification of the Fejér–Hermite polynomials in the form $\Phi_n(f, x) = F_n\left[f(\lambda_n t); \frac{1}{\lambda_n} x\right]$:

$$\Phi_n(f, x) = \frac{1}{n^2} \sum_{\nu=1}^{n} \left(1 - \frac{a_\nu^{(n)} x}{\lambda_n}\right) \left[\frac{T_n(x/\lambda_n)}{(x/\lambda_n) - a_\nu^{(n)}}\right] f\left(\lambda a_\nu^{(n)}\right).$$

Denote $\exp^1 |x| = e^{|x|}$, $\exp(\exp^m |x|) = \exp^{m+1} |x|$ (m a positive integer). Then the growth order $f(x) = O(\exp^m |x|)$ ($|x| \to \infty$) is suitable for describing any type of growth of $f(x)$ as $|x| \to \infty$, if m is chosen arbitrarily. Similarly to the preceding, set

$$\log n = \log^1 n, \qquad \log(\log^{(m)} n) = \log^{(m+1)} n.$$

Theorem 5. Let $0 < \theta < \dfrac{1}{m+2}$. Then for every continuous function $f(x)$, defined on $(-\infty, +\infty)$, with the condition $l(x) = O(|x|^m)$, we have:

$$\lim_{n\to\infty} F_n[f(n^\theta t),\, n^{-\theta} x] = f(x) \qquad (-\infty < x < \infty). \tag{6}$$

Moreover, (6) holds almost uniformly on $(-\infty, +\infty)$.

Theorem 6. For every continuous function $f(x) = O(\exp^m |x|)$ ($|x| \to \infty$), we have

$$\lim_{n\to\infty} F_n \left[f\left(t \log^{m+1}(n)\right);\, \frac{x}{\log^{m+1}(n)}\right] = f(x).$$

Moreover, (7) holds almost uniformly on $(-\infty, +\infty)$.

The proof of these two theorems is entirely analogous in principle to the proof of Theorem 2 (or 3). Indeed, $F_n(f, x)$ is a linear positive operator, so that the main idea of Theorem 1 (with slight changes) can again be applied to the present case with $\Omega(|x|) = |x|^m$, $\Omega(|x|) = \exp^m |x|$.

Theorems 5 and 6 can also be proved by more direct methods, by means of an appropriate modification and generalization of Fejér' s original proof.

Remark 4. Theorem 6 shows that the generalization of the Fejér–Hermite polynomial operators can be used for the approximation of continuous functions of any order of growth on the real axis. This appears somewhat unexpected when compared with the well-known polynomials of Levitan [7].

Finally, as is known from Bernstein' s results, the rate of convergence of Fejér polynomials is of order $O(1) \cdot \omega_f \left(\frac{\log n}{n}\right)$, if $f(x)$ is continuous on $[-1, 1]$. One may suppose that the generalization of the Fejér–Hermite polynomials has better convergence than the generalizations of the Landau polynomials on $(-\infty, +\infty)$. But we are not yet able to substantiate this supposition. The most interesting conclusion that can be drawn from Theorem 6 is that the unboundedness of a continuous function being approximated on $(-\infty, +\infty)$ creates no difficulties in approximating it by methods of increasing multipliers.

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Note: Figure translations are in progress. See original paper for figures.