MATHEMATICS

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1964

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Abstract

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MATHEMATICS

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APPROXIMATIVE PROPERTIES OF SUBSPACES OF FINITE DEFECT IN THE SPACE OF CONTINUOUS FUNCTIONS

(Presented by Academician A. N. Kolmogorov, 29 XI 1963)

Let $\mathcal{L}^n$ be a subspace of defect $n < \infty$ in the space $C(T)$ of continuous functions $x(t)$ defined on a bicompact space T. We shall say that the subspace $\mathcal{L}^n$ has property E (respectively U) if for every element $x \in C(T)$ there exists in the subspace $\mathcal{L}^n$ a nearest element (respectively, no more than one such element). A subspace possessing property E and property U is called **Chebyshev**. Subspaces $\mathcal{L}^n$ with the indicated properties were studied in the papers ([1–5]). Here we give general criteria for such subspaces, solving, in particular, the problems posed in ([3,4]). A subspace $\mathcal{L}^n \subset C(T)$, as is known, is given by a system of equations

$$\int_T x(t)\, d\mu_i(t) = 0 \qquad (i = 1, \dots, n),$$

where $\mu_i(e)$ are bounded regular measures on the class of Borel sets $\{e\}$ of the bicompact space T. The linear combinations $\mu = a_1\mu_1 + \dots + a_n\mu_n$ constitute the annihilator M_n of the subspace $\mathcal{L}^n$. We shall use the following propositions, which follow from general theorems established by the author in ([5]).

A. The subspace $\mathcal{L}^n \subset C(T)$ has property E (respectively U) if and only if, for every functional $f(\mu)$ defined on the annihilator M_n, there exists an element $x(t) \in C(T)$ (respectively, no more than one such element) such that

$$\|x\| = \|f\|; \qquad f(\mu) = \int_T x(t)\, d\mu(t) \quad (\mu \in M_n). \tag{1}$$

B. If the subspace $\mathcal{L}^n$ has property E, then for every measure $\mu \in M_n$ there exists a maximal element $x \in C(T)$, i.e. an element x such that

$$\|x\| = 1; \qquad \int_T x(t)\, d\mu(t) = \operatorname{var}(\mu, T) = \|\mu\|. \tag{2}$$

C. The subspace $\mathcal{L}^n$ has property U if and only if, for every $\mu \in M_n$, the dimension r of the set of maximal elements does not exceed $n-1$, and there exist $r+1$ maximal elements $x_0, x_1, \dots, x_r$, for which the rank of the matrix

$$\left\| \int_T x_k(t)\, d\mu_i(t) \right\|_{k=0,\ i=1}^{k=r,\ i=n}$$

is equal to $r+1$.

By $S(\mu)$, $S(\mu)^i$, and $S(\mu)^*$ we denote, respectively, the support of the measure μ, the set of its isolated points, and the set of its limit points.

Theorem 1. *The subspace $\mathcal{L}^n \subset C(T)$ has property E if and only if: a) for every measure $\mu \in M_n$ there exist closed sets $S(\mu)^+$ and $S(\mu)^-$, forming the μ-decomposition of the support $S(\mu)$ into*

in the sense of Hahn; b) for any measures $\mu, \bar{\mu} \in M_n$ the set $S(\mu) \setminus S(\bar{\mu})$ is closed; c) the measure μ is absolutely continuous with respect to ν on the set $S(\mu)$.

We shall set out the main points of the proof.

Sufficiency. Put $\nu(e) = \mathrm{var}(\mu_1, e) + \dots + \mathrm{var}(\mu_n, e)$ and consider the space M of measures $\{\mu\}$ that are absolutely continuous with respect to the measure ν. Clearly, $M_n \subset M$. Let $f(\mu)$ be a functional on M_n, $\|f\| = 1$, $f(\mu_0) = \|\mu_0\| = \mathrm{var}(\mu_0, T)$ ($\mu_0 \in M_n$), and let $F(\mu)$ be its minimal extension to all of M (so that $\|F\| = \|f\| = 1$). From the Radon-Nikodym theorem it follows that $F(\mu)$ can be represented in the form

$$F(\mu) = \int_T \alpha(t) \varphi_\mu(t)\, d\nu(t) = \int_T \alpha(t)\, d\mu(t) = \int_{S(\nu)} \alpha(t)\, d\mu(t),$$

where $\alpha(t)$ is an essentially (ν) bounded function, $\varphi_\mu(t)$ is a summable (ν) function, $d\mu(t) = \varphi_\mu(t)\, d\nu(t)$, and

$$\|F\| = \mathrm{vrai\,max}_{(\nu)\,T} |\alpha(t)| = 1.$$

Since

$$F(\mu_0) = f(\mu_0) = \mathrm{var}(\mu_0, T) = \int_T |\varphi_{\mu_0}(t)|\, d\nu(t),$$

we have $\alpha(t)\varphi_{\mu_0}(t) = |\varphi_{\mu_0}(t)|$ almost everywhere (ν) on $S(\nu)$. Hence it is not hard to obtain that

$$\alpha(t) = \begin{cases} +1, & \text{if } t \in S(\mu_0)^i \cap S(\mu_0)^+, \\ -1, & \text{if } t \in S(\mu_0)^i \cap S(\mu_0)^-, \end{cases}$$

$$\alpha(t) = \begin{cases} +1 & \text{almost everywhere } (\mu_0), \text{ if } t \in S(\mu_0)^* \cap S(\mu_0)^+, \\ -1 & \text{almost everywhere } (\mu_0), \text{ if } t \in S(\mu_0)^* \cap S(\mu_0)^-. \end{cases} \tag{3}$$

Requiring that the equalities (3) hold everywhere on $S(\mu_0)^*$, we make $\alpha(t)$, by condition a), continuous on the set $S(\mu_0)$. At the same time, thanks to condition c), we shall still have

$$f(\mu) = \int_T \alpha(t)\, d\mu(t) = \int_{S(\nu)} \alpha(t)\, d\mu(t) \qquad (\mu \in M_n). \tag{4}$$

Put $T' = S(\nu) \setminus S(\mu_0)$. From condition b) it is easy to infer that the set T' is closed. Next we have

$$f(\mu) = \int_{S(\nu)} \alpha(t)\, d\mu(t) = \int_{S(\mu_0)} \alpha(t)\, d\mu(t) + \int_{T'} \alpha(t)\, d\mu(t) \qquad (\mu \in M_n). \tag{5}$$

Consider the functional

$$f'(\mu') = \int_{T'} \alpha(t)\, d\mu(t)$$

on the subspace M'_n, formed by the restrictions μ' of the measures $\mu \in M_n$ to the bicompact set T'.

Two cases are possible.

First case. $f'(\mu') < \operatorname{var}(\mu', T')$ (or $f'(\mu') = 0$) for every measure $\mu' \in M'$. Then $\|f'\| = c < 1$, and, by the well-known Helly theorem from the theory of moments, there exists a function $\alpha'(t) \in C(T')$, continuous on T', such that

$$\|\alpha'\| \leq c + \varepsilon < 1, \qquad f'(\mu') = \int_{T'} \alpha'(t)\, d\mu(t) \qquad (\mu' \in M'_n).$$

Thus, taking (5) into account, we may regard the function $\alpha(t)$ in (4) as continuous also on the closed set $S(\mu_0) \cup T'$. Extending it by Urysohn' s theorem to the whole bicompact space, we obtain a continuous function $x(t)$ satisfying the relations (1).

Second case. For some $\mu' \in M_n$, $|f'(\bar{\mu}')| = \operatorname{var}(\bar{\mu}', T')$. Then, proceeding in the same way as in the case of the measure μ'_0, we see that $\alpha(t)$ may be assumed continuous on the set $S(\bar{\mu}')$. Setting further $T'' = T' \setminus S(\bar{\mu}')$, we again consider two possible cases analogous to those indicated above. It is easy to understand that in a finite number of steps we shall construct a function $x(t) \in C(T)$ satisfying relations (1).

Necessity. Let $\mathcal{L}^n$ have property E, let $\mu \in M_n$, and let $x(t)$ be a function satisfying (2). It is not difficult to see that $|x(t)| = 1$ for $t \in S(\mu)$, and that the sets $S(\mu)^+ = \{t \in S(\mu) : x(t) = +1\}$ and $S(\mu)^- = \{t \in S(\mu) : x(t) = -1\}$ satisfy condition a). Suppose now that condition a) is satisfied, but condition b) is violated. Then, taking into account the closedness of the supports, it is not difficult to show that there exists a point

$$t_0 \in S(\mu) \cap [\overline{S(\mu)} \setminus S(\mu)].$$

For definiteness, we shall assume that

$$t_0 \in S(\mu)^+ \cap S(\bar{\mu})^+.$$

Consider on the annihilator M_n the functional

$$f(\mu') = \int_{S(\mu)\cup S(\bar{\mu})} \alpha(t)\, d\mu'(t) \qquad (\mu' \in M_n), \tag{6}$$

where

$$\alpha(t) = \begin{cases} +1, & \text{if } t \in S(\mu)^+ \cup [S(\bar{\mu})^- \setminus S(\mu)], \\ -1, & \text{if } t \in S(\mu)^- \cup [S(\bar{\mu})^+ \setminus S(\mu)]. \end{cases}$$

Taking condition a) into account, one can show that for this functional there is no function $x(t) \in C(T)$ satisfying relations (1). Suppose now that, while condition a) is satisfied, condition c) is not satisfied. Then there exists a set $e \subset S(\bar{\mu})^*$ such that $\operatorname{var}(\bar{\mu}, e) = 0$, but $\mu(e) \neq 0$. We shall assume that

$$e \subset S(\mu)^+ \cap S(\bar{\mu})^+,$$

and construct the function

$$\alpha(t) = \begin{cases} +1, & \text{if } t \in [S(\bar{\mu})^+ \setminus e] \cup [S(\bar{\mu})^- \setminus S(\bar{\mu})], \\ -1, & \text{if } t \in S(\bar{\mu})^- \cup [S(\mu)^+ \setminus S(\bar{\mu})] \cup e. \end{cases}$$

It can be shown that, with this definition of $\alpha(t)$, the functional (6) also cannot be expressed in the form (1).

Theorem 2. The subspace $\mathcal{L}^n \subset C(T)$ has property U if and only if, for every measure $\mu \in M_n$ satisfying condition a) of Theorem 1, the following conditions hold: 1) the number of points $t_1, \dots, t_r$ of the set $T \setminus S(\mu)$ does not exceed $n-1$; 2) the rank of the matrix $\|\mu_i(t_k)\|_{i=1,k=1}^{i=n,k=r}$ is equal to r.

This theorem can be derived from the proposition B stated above.

From Theorems 1 and 2 it follows that

Theorem 3. The subspace $\mathcal{L}^n \subset C(T)$ is Chebyshev if and only if, for every measure $\mu \in M_n$, the following conditions hold: 1) the set $T \setminus S(\mu)$ contains no more than $n-1$ points. 2) For any r isolated points $t_1, \dots, t_r$ $(r \leq n-1)$ of the bicompactum T, the rank of the matrix $\|\mu_i(t_k)\|_{i=1,k=1}^{i=n,k=r}$ is equal to r. 3) There exist closed sets T^+ and T^- forming a μ-decomposition of the bicompactum T. 4) μ and μ_1 are mutually singular on the set T^* of limit points of the bicompactum T.

If the bicompactum T contains at least n isolated points, then condition 2) may be omitted.

In the case when T is a countable bicompactum and $n > 1$, condition 4) is equivalent to the assertion that $\mu(t) = 0$ for $t \in T^*$ and any $\mu \in M_n$. In particular cases, for example for the space c of convergent sequences, the conditions of the theorems presented simplify considerably.

The criteria established make it possible to study the question of the existence of subspaces possessing the properties E and U, depending on the topological structure of the bicompactum T. In this direction the following results have been obtained.

1. On every bicompactum T (i.e., in the space $C(T)$) there exist subspaces of any finite defect possessing property E (respectively, property U)*.
2. If on a bicompactum T there exists a Chebyshev subspace of finite defect, then the bicompactum T contains at most a countable number of pairwise disjoint open sets.
3. Chebyshev subspaces of unit defect exist on every compactum.
4. Chebyshev subspaces of finite defect different from one can exist only on bicompacta containing no infinite connected open set**.
5. If a compactum T coincides with the closure of its isolated points (in particular, is countable), then on it there exist Chebyshev subspaces of any finite defect.

A compactum of the indicated type may even contain an infinite connected closed set (a continuum)***. An example is the closure of the set of points in the plane of the form

$$\left\{\frac{n}{2^k}, \frac{1}{2^k}\right\} \quad (n = 0, 1, \dots, 2^k;\ k = 0, 1, \dots).$$

Let us point out, in conclusion, the connection that exists between the questions considered and the following problem from the theory of moments (going back to Helly). On a bicompactum T there are given n regular measures $\mu_1, \dots, \mu_n$ and numbers $c_1, \dots, c_n$. It is required to find a continuous function $x(t) \in C(T)$

of the least possible norm which would satisfy the conditions

$$\int_T x(t)\, d\mu_i(t) = c_i \quad (i = 1, \dots, n).$$

By virtue of Proposition A, the criteria established in Theorems 1, 2, and 3 are simultaneously conditions for the existence and uniqueness of the solution of this problem for an arbitrary set of numbers $c_1, \dots, c_n$.

The author expresses his gratitude to S. B. Stechkin for discussion of the work at the seminar he directed.

Received29 XI 1963

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* For the sake of brevity in the formulations, the bicompactum T here and below is regarded as infinite.

** In essence this was proved in the second part of Theorem 11 of (5), whose formulation contains an error.

*** This contradicts the proposition formulated in (4), according to which Chebyshev subspaces of finite defect $n > 1$ can exist only on totally disconnected bicompacta.

Note: Figure translations are in progress. See original paper for figures.