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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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EQUILIBRIUM OF A PLASMA COLUMN WITH HELICAL PERTURBATIONS

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In studying the equilibrium ([1]) and stability ([2])* of toroidal plasma configurations, configurations with a system of nested magnetic surfaces are usually chosen as the unperturbed state. In this case the linearized equations for perturbations have singularities on magnetic surfaces containing closed lines of force, so that, as the perturbation frequency $\omega \to 0$, the solution tends to infinity in a neighborhood of these surfaces. We shall show below that the appearance of singularities is connected with a qualitative restructuring of the magnetic-surface structure, and that the presence of arbitrarily small static perturbations leads to finite changes in plasma configurations that are essential for their stability.

Fig. 1

1. Cylindrical magnetic surfaces with helical perturbations. Let us first consider static perturbations of a system of nested cylindrical surfaces ($r =$ const) of a magnetic field $\mathbf{B} = \overline{\mathbf{B}}(r)$ in the general case, without assuming that equilibrium conditions are satisfied. If the perturbations have helical symmetry, so that now the field $\mathbf{B} = \overline{\mathbf{B}}(r) + \widetilde{\mathbf{B}}(r, \theta)$, then the perturbed magnetic surfaces also possess helical symmetry and in cylindrical coordinates r, φ, z are described by the equation $\psi(r, \theta) = A_z + \alpha r A_\varphi =$ const. Here $\mathbf{B} = \operatorname{rot} \mathbf{A}$, $\theta = \varphi - \alpha z$, $L = 2\pi/\alpha$ is the pitch of the perturbation helix, and the function ψ is equal to ([3])

$$\psi = \int_0^r (\alpha r \overline{B}_z - \overline{B}_\varphi)\, dz + \widetilde{\psi} = \int_0^r (\alpha - \overline{\mu})\, \overline{B}_z r\, dr + f(r) \cos m\theta, \qquad (1)$$

where $\overline{\mu} = \overline{B}_\varphi / r\overline{B}_z$, and as the perturbation $\widetilde{\psi}(r, \theta)$ an m-fold harmonic of the helical magnetic field has been chosen. The equality $\overline{\mu} = \alpha$ means that the pitch of the lines of force of the unperturbed field $\overline{\mathbf{B}}$ coincides with the pitch of the perturbation helix, and determines the radii r_s of cylinders which in the unperturbed state are formed by "closed" (i.e., periodic with period L) lines of force. It is easy to see that near $r = r_s$, for a sufficiently arbitrary function $f(r)$, a wave-like structure of magnetic surfaces is generally formed, as shown in Fig. 1. Indeed, setting the derivatives ψ_r and ψ_θ equal to zero, we find the coordinates of the singular points: $m\theta = 0, \pi, 2\pi, \dots, (2m-1)\pi$ and $(\alpha - \overline{\mu})\overline{B}_z r \pm f' = 0$. The type of singular point is determined by the sign of the invariant $\Sigma \equiv \psi_{rr}\psi_{\theta\theta} - \psi_{r\theta}^2 \simeq \pm m^2 r f \overline{B}_z \mu'$. Elliptic and hyperbolic singular points (for small f) lie on the circles $r = r_s \pm f'/\mu'\overline{B}_z r_s$. The equation of the separatrix of the fibers, accurate up to terms–

* In ([1],[2]) references to the corresponding literature may be found.

is $\sim f^{1/2}$, i.e., $r = \bar{r}_s \pm (2f\cos m\theta/\mu'\bar{B}_z r_s)^{1/2}$. The appearance of fibers becomes evident if we note that the function $\bar{\psi}(r)$ has a maximum at $r = \bar{r}_s$, and, consequently, the profile $\psi(r,\theta)$ under a perturbation periodic in θ will, generally speaking, have a series of alternating maxima and minima in θ. The example considered clearly shows that magnetic surfaces with closed field lines are degenerate ([3],[4]), and this degeneracy can be removed, in particular, by a **helical** perturbation, so that the resulting magnetic surfaces with fibers are described by an exact integral of the field-line equations $\psi(r,\theta) = \text{const}$. Since the arguments given did not assume that equilibrium conditions are satisfied, they are applicable, for example, to a vacuum field: $\bar{B}_z = \text{const}$, $\bar{B}_\varphi = \text{const}/r$, $f(r) = brI'_m(amr)$.

2. **Equilibrium helical plasma configurations.** Let us now consider an equilibrium plasma configuration consisting of a circular cylinder with distributed currents, with a small helical deformation of its surface. From the equilibrium equation $4\pi p = [\mathbf{jB}]$ ($\mathbf{j} = \operatorname{rot}\mathbf{B}$), under the condition of helical symmetry, there follow the equalities

$$\frac{\partial(p,\psi)}{\partial(r,\theta)} = 0, \qquad \frac{\partial(I,\psi)}{\partial(r,\theta)} = 0, \tag{2}$$

where $I \equiv B_z + \alpha r B_\varphi$. It follows from (2) that p and I are functions of ψ; moreover, if $p = p(\psi)$ and $I = I(\psi)$, then equations (2) are satisfied everywhere where the derivatives $dp/d\psi$ and $dI/d\psi$ exist.

Using the functions $p(\psi)$ and $I(\psi)$, we can represent the equilibrium equation in the form of a single equation for the function ψ ([5],[7])

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{r}{\beta}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2\alpha I}{\beta^2} + \frac{1}{2\beta}\frac{dI^2}{d\psi} + 4\pi\frac{dp}{d\psi} = 0, \tag{3}$$

where $\beta \equiv 1 + \alpha^2 r^2$. Suppose that in the unperturbed state there exists a magnetic surface $r = \bar{r}_s$, where $\partial\bar{\psi}/\partial r = 0$, i.e. $\alpha = \mu$. We shall show that the equilibrium conditions for the perturbed state necessarily lead either to splitting of the magnetic surfaces and the formation of fibers, or to the fact that on the magnetic surface with $\partial\psi/\partial n = 0$* one also has $\partial p/\partial n = 0$ and $\partial I/\partial n = 0$.

Indeed, the component of the current parallel to the helical line is proportional to

$$j_z + \alpha r j_\varphi = \frac{1}{2}\frac{dI^2}{d\psi} + 4\pi\beta\frac{dp}{d\psi}.$$

On the magnetic surface with $\partial\psi/\partial n = 0$, if $\partial p/\partial n$ and $\partial I/\partial n$ are not equal to zero, both terms on the right-hand side are infinite. In the unperturbed state $\psi = \bar{\psi}(r)$ these terms compensate one another. However, in the case of a substantial dependence $r = r(\theta)$, such compensation at finite currents is impossible. The magnetic surface on which $\partial\psi/\partial n = 0$ also cannot remain circular, since the boundary conditions on it, $\psi = \text{const}$, $\partial\psi/\partial r = 0$, lead to an axially symmetric solution of equation (3), which contradicts the assumption of a helical deformation of the boundary. Thus, if the spatial derivatives of p and I do not vanish, then special points of the family $\psi(r,\theta) = \text{const}$ necessarily appear.

Let us expand $\psi(r,\theta)$ in a neighborhood of a special point $r = r_s$, $\theta = 0$ (where $\psi_r = \psi_\theta = 0$):

$$\psi = \psi_0 + \psi_{rr}\rho^2/2 + \psi_{\theta\theta}\theta^2/2 + \psi_{r\theta}\rho\theta + ...$$

Here $\rho = r - r_s$, and θ is measured from the special point. If we substitute the expansions

$$p = p_0 + p_r\rho + p_\theta\theta + p_{rr}\rho^2/2 + p_{\theta\theta}\theta^2/2 + p_{r\theta}\rho\theta + ...$$

and

$$I = I_0 + I_r\rho + I_\theta\theta + I_{rr}\rho^2/2 + I_{\theta\theta}\theta^2/2 + I_{r\theta}\rho\theta + ...$$

into equations (2) and equate the linear terms, we find that either

$$\Sigma \equiv \psi_{rr}\psi_{\theta\theta} - \psi_{r\theta}^2 = 0,$$

or

$$p_r = p_\theta = I_r = I_\theta = 0.$$

The equality $\Sigma = 0$ means that the special point is degenerate, and we have a family of nested magnetic surfaces. It was shown above

* n is the normal to the section of the magnetic surface by the plane $z = \text{const}$.

it has been shown that such a situation can occur only if, on the singular magnetic surface, $p_r = I_r = 0$. Consequently, in all cases the first derivatives of p and I are zero, and this means that in the presence of arbitrarily small field perturbations at the extremum of the pressure distribution $p(r)$ (and also $I(r)$)

"steps"are formed, where $p'(r) = 0$ (and $I'(r) = 0$). From equating the quadratic terms in (2) there follow the equalities

$$p_{rr}/\psi_{rr} = p_{\theta\theta}/\psi_{\theta\theta} = p_{r\theta}/\psi_{r\theta} = \nu = \text{const}, \qquad I_{rr}/\psi_{rr} = I_{\theta\theta}/\psi_{\theta\theta} = I_{r\theta}/\psi_{r\theta} = \varepsilon = \text{const}.$$

Therefore, in the neighborhood of a singular point $p = p_0 + \nu\psi + \cdots$, $I = I_0 + \varepsilon\psi + \cdots$, and, consequently, there exists a regular solution of equation (3), which can be represented in the form $\psi = \psi(r) + f_m(r)\cos m\theta$. With such a choice of the perturbation, singular points of elliptic and hyperbolic type of the family $\psi(r, \theta) = \text{const}$ necessarily appear.

3. **On Suydam' s local stability criterion.** Linearized with respect to $\widetilde{\psi} = f_m(r)\cos m\theta$, equation (3) has the form

$$\left(\frac{rf_m'}{\beta}\right)' - \left\{\frac{m^2}{r} + \frac{1}{r^2 J}\left(\frac{r^3 J'}{\beta}\right)' + \frac{4\alpha^2 B_\varphi}{\beta^2 J} + \frac{8\pi\alpha^2 p'}{\beta J^2}\right\} f_m = 0, \qquad (4)$$

where $J = aB_z - B_\varphi/r = (a - \mu)B_z$. This equation coincides with the equation obtained by Suydam ([6]) for the radial displacement $\xi_r \sim \widetilde{\psi}/rJ$ at frequency $\omega = 0$. In accordance with what has been said, it has a singular point at $J = 0$ ($\mu = a$), in whose neighborhood, for $p'(r) \neq 0$, no bounded solution exists. In connection with the latter, let us note once more that the functions $p(\psi)$ and $I(\psi)$, generally speaking, change under static deformation of the equilibrium configuration, and, if the cylindrical column is regarded as the limit of a perturbed column, then at the points where $J = 0$ one must also have $p' = 0$.

In studying plasma stability, the question arises of the choice of the equilibrium configuration. This choice is sufficiently arbitrary and is usually determined by the simplicity of the solution of the problem. In particular, Suydam' s local stability criterion ([6]),

$$32\pi p' + rB_z^2(\mu'/\mu)^2 > 0,$$

was obtained from an analysis of equation (4) in the neighborhood of the singular point $J = 0$. In this case, the configuration with cylindrical magnetic surfaces and with $p'(r) < 0$ was taken as the initial equilibrium configuration. However, as we have seen above, in the presence of arbitrarily small static perturbations (which can be expanded in helical harmonics) the equilibrium configuration undergoes substantial changes precisely in the neighborhood of the singular point $J = 0$. Therefore, if as the initial equilibrium configuration we choose a cylindrical plasma column, regarded as the limit of a perturbed column for a perturbation tending to zero, then, since $p'(r_s) = 0$, the local plasma stability criterion reduces simply to the condition for the stability of the magnetic field, $\mu' \neq 0$, in the neighborhood of a closed field line ([8]).

It is evident from this that the question of the choice of the initial equilibrium configuration is not trivial. In particular, the process of development of instability may reduce simply to the formation of a stable fibrous structure of the plasma ([3]).

4. **Helical equilibrium plasma configurations for linear functions $p(\psi)$ and $I(\psi)$.** Let us consider, as an example, equilibrium configurations for $p(\psi) = p_0 + \nu\psi$, $I(\psi) = I_0 + \varepsilon\psi$. In this case equation (4) is exact and can be solved in Bessel functions. The solutions of the unperturbed equation (3) and of the equation for the helical perturbation $\widetilde{\psi} = f_m(r)\cos m\theta$ for $\varepsilon \neq 0$

have, respectively, the form

$$\bar{\psi} = A_0\left[J_0(\varepsilon r) + \alpha r J_1(\varepsilon r)\right] - \frac{I_0}{\varepsilon} - \frac{\nu}{\varepsilon^2}\left(\beta + \frac{2\alpha}{\varepsilon}\right), \tag{5}$$

$$f_m = A_1\left[\varepsilon J_m(\chi r) - \alpha r \chi J_m'(\chi r)\right], \qquad \chi^2 = \varepsilon^2 - \alpha^2 m^2. \tag{6}$$

If, however, $\varepsilon = 0$, then the solutions will be $\bar{\psi} = I_0\alpha r^2/2 - \nu\beta^2/8\alpha^2$, $f_m = b_m r I_m'(\alpha m r)$. In this case the unperturbed pressure distribution has a minimum at $r = 0$ and one maximum at $r = \bar{r}_s$. If $\varepsilon \neq 0$, the unperturbed fields are $B_\varphi = b_{\varphi 0} r + B_{z1} J_1(\varepsilon r)$ and $B_z = B_{z0} + B_{z1} J_0(\varepsilon r)$, where $\varepsilon = 2\alpha\mu_0/(\alpha - \mu_0)$, $\mu_0 = b_{\varphi 0}/B_{z0}$. They represent a superposition of a uniform longitudinal field, the azimuthal field of a uniform current, and fields of a force-free equilibrium configuration. In this case the equation $\partial\bar{\psi}/\partial r = 0$ has a number of solutions $r = \bar{r}_{sn}$, in the neighborhood of which fibers are formed.

Let us also note that for discrete values $\alpha = \alpha_{mn}$, determined by the equality $\chi a = x_{mn}$, where x_{mn} are the roots of $f_m(\chi r)$, we have a helical structure of the plasma column with the cylindrical surface $r = a$.

5. **Stability of a force-free plasma configuration.** Particular cases of (5) are the case of a uniform current in a uniform longitudinal field ($B_{z1} = 0$) and the case of a force-free equilibrium configuration ($B_{z0} = b_{\varphi 0} = 0$).

In the first case, static helical perturbations with period $L = 2\pi/\mu_0$ completely destroy the equilibrium configuration ([8]), all magnetic surfaces of which consist of closed field lines.

In the second case of a force-free equilibrium configuration, the existence of the solution (6) makes it possible to obtain the stability condition for a plasma supported by an ideally conducting shell located at $r = a$. Indeed, following B. B. Kadomtsev ([2]), we conclude that the plasma is stable if $\chi^2 a^2 < x_{11}^2$, where $x_{11} \neq 0$ is the smallest root of $f_m(\chi r)$. In our case $\varepsilon = j/B$, $\alpha m = k = 2\pi/\lambda$, and the quantity $\chi^2 = \varepsilon^2 - k^2$. For small $k^2 a^2$, the stability condition will be $j/B < x_{11}/a$, where x_{11} is close to the first root of $J_1(x)$. This criterion can be applied to a plasma column closed into a torus of radius R, if $a^2/R^2 \ll 1$. In the same notation, the Shafranov-Kruskal criterion ([2]) has the form $j_z/B_z < 2/R$, from which it follows that the force-free configuration permits currents approximately R/a times larger.

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Note: Figure translations are in progress. See original paper for figures.