



Soviet-era science, translated into English

MATHEMATICS

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.78716>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

A. I. Perov

On a Multidimensional Linear Differential Equation of the Second Order

(Presented by Academician A. Yu. Ishlinsky on 10 VI 1964)

1. Let E_x be a real m -dimensional space. We denote by $L_p(E_x) = L_p$ the linear space of all p -linear functionals defined on E_x . We denote by $S_p(E_x) = S_p$ the subspace of this space consisting of all p -linear symmetric functionals.

Consider, in a domain G of the space E_x , the multidimensional linear differential equation of the second order

$$\mathcal{L}\{y(x)\} \equiv y''(x) + P(x)y'(x) + Q(x)y(x) = f(x) \quad (1)$$

under the following assumptions: for each fixed $x \in G$, the operator $P(x)$ is a linear operator from the space L_1 into the space L_2 ; $Q(x)$ and $f(x)$ are elements of the space L_2 .

If a coordinate system x^1, \dots, x^m is introduced in the space E_x , then equation (1) is written in the form of the system

$$\mathcal{L}_{ij}\{y(x)\} \equiv \frac{\partial^2 y(x)}{\partial x^i \partial x^j} + \sum_{k=1}^m p_{kij}(x) \frac{\partial y(x)}{\partial x^k} + q_{ij}(x)y(x) = f_{ij}(x). \quad (1^*)$$

Here $i, j = 1, 2, \dots, m$.

By a solution of equation (1) we mean a function $y(x)$, defined in a domain $G^* \subset G$, possessing continuous first and second derivatives and satisfying equation (1) at every point $x \in G^*$.

A particular solution of equation (1) may be singled out either by means of initial data

$$y(\xi) = \eta, \quad y'(\xi) = \zeta \quad (\xi \in G, \zeta \in L_1) \quad (2)$$

(the Cauchy problem), or by means of conditions of the form

$$y(\xi_i) = \eta_i \quad (\xi_i \in G; i = 1, 2, \dots, m+1) \quad (3)$$

(the $(m + 1)$ -point problem). If, for arbitrary initial data (2), equation (1) has a unique solution, then we shall say that equation (1) is **completely solvable**. In the present paper we set forth some results obtained in the investigation of completely solvable equations (a number of results concerning the general theory of linear equations of the first order may be found in ⁽¹⁻³⁾).

Throughout the paper it is assumed that the coefficients $P(x)$ and $Q(x)$ and the free term $f(x)$ are continuous in the domain G .

2. The Cauchy problem. We first formulate a uniqueness theorem.

Theorem 1. *Let $y_1(x)$ and $y_2(x)$ be two solutions of equation (1), satisfying the same initial conditions (2) and defined on the domains G_1 and G_2 , respectively.*

Then $y_1(x) \equiv y_2(x)$ on that connected component of the set $G_1 \cap G_2$ which contains the point ξ .

(We note that this theorem is not a consequence of the analogous theorem for an ordinary second-order equation.)

For continuous $P(x)$, $Q(x)$, and $f(x)$, the Cauchy problem (1)–(2) may have no solutions (for $m > 1$). However, if it is additionally known that $P(x)$, $Q(x)$, and $f(x)$ are continuously differentiable in the domain G , then we arrive at the following theorem.

Theorem 2 (cf. with (1, 3)). *For the complete solvability of equation (1) it is necessary and sufficient that the following conditions be satisfied:*

a) $P(x)\xi$, $Q(x)$, and $f(x)$ are symmetric bilinear functionals ($x \in G$, $\xi \in L_1$), i.e. $\in S_2$.

b)

$$\bigwedge (-P(x)[f(x)h]k + f'(x)hk) = 0,$$

$$\bigwedge (P(x)[Q(x)h]k - Q'(x)hk) = 0,$$

$$\bigwedge (-[P'(x)h]\xi k + P(x)[P(x)\xi h]k - Q(x)k(\xi h)) = 0$$

for all $\xi \in L_1$ and $h, k \in E_x$. (Here the symbol \bigwedge has the following meaning:

$$\bigwedge Ahk = \frac{1}{2}(Ahk - Akh).$$

If the domain G is simply connected, then every solution can be extended to the whole domain G . But if the domain G is not simply connected, then one can give examples of equations for which some solutions cannot be extended to the whole domain G with uniqueness preserved.

3. Homogeneous equation. Many facts of the general theory of ordinary second-order equations can be carried over to multidimensional equations.

Let $y_1(x), \dots, y_k(x)$ be a system of continuously differentiable functions defined in the domain G . The matrix

$$W(x) = W \left\{ \begin{matrix} y_1, \dots, y_k \\ x^1, \dots, x^m \end{matrix} \right\} = \begin{pmatrix} y_1(x) & y_2(x) & \dots & y_k(x) \\ \frac{\partial y_1(x)}{\partial x^1} & \frac{\partial y_2(x)}{\partial x^1} & \dots & \frac{\partial y_k(x)}{\partial x^1} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial y_1(x)}{\partial x^m} & \frac{\partial y_2(x)}{\partial x^m} & \dots & \frac{\partial y_k(x)}{\partial x^m} \end{pmatrix} \quad (4)$$

will be called the **Wronski matrix**. The determinant of this matrix for $k = m + 1$ will be called the **Wronskian determinant**. We note the following result.

Theorem 3. *If the system $y_1(x), \dots, y_k(x)$ is linearly dependent, then the rank of the Wronski matrix is $< k$.*

If $y_1(x), \dots, y_k(x)$ is a linearly independent system of solutions of the homogeneous equation (1) ($f(x) \equiv 0$), then the rank of the Wronski matrix is identically equal to k .

A system of $(m + 1)$ linearly independent solutions of the (homogeneous) equation (1) will be called a (complete) **fundamental system**. It is not difficult to see that, under the conditions of Theorem 2, the homogeneous equation has a complete fundamental system of solutions, and the general solution of the homogeneous equation has the form

$$y(x) = c^1 y_1(x) + \dots + c^{m+1} y_{m+1}(x). \quad (5)$$

From a fundamental system of solutions the differential operator \mathcal{L} is uniquely reconstructed:

$$\mathcal{L}_{ij}\{y(x)\} = \frac{\left| \begin{array}{c|c} W(x) & \begin{matrix} \frac{y(x)}{dx^1} \\ \vdots \\ \frac{\partial y(x)}{\partial x^m} \end{matrix} \\ \hline \frac{\partial^2 y_1}{\partial x^i \partial x^j} & \dots & \frac{\partial^2 y_{m+1}}{\partial x^i \partial x^j} & \frac{\partial^2 y(x)}{\partial x^i \partial x^j} \end{array} \right|}{\det W(x)}, \quad (6)$$

whence expressions for $p_{kij}(x)$ and $q_{ij}(x)$ can be found simply. The Ostrogradsky-Liouville formula in our case takes the form (cf. (4))

$$\det W(x) = \exp \left\{ \int_{\xi}^x \left\{ - \sum_{s=1}^m \sum_{k=1}^m p_{kks}(t) \right\} dt \right\} \det W(\xi). \quad (7)$$

Let us present some results concerning the study of the structure of the set of zeros of solutions of the second-order equation. Let $\mathfrak{M} = \{x \in E_x : y(x) = 0\}$ and $\xi \in \mathfrak{M}$. By the uniqueness theorem $y'(\xi) \neq 0$, and therefore the point ξ is a regular point of the manifold \mathfrak{M} . Each element $x \in E_x$ can be represented uniquely in the form $x = \xi + u + \alpha v$, where $u \in U = \{h : y'(\xi)h = 0\}$ and $v = y'(\xi)$ (see (5)), and our manifold \mathfrak{M} in a neighborhood of the point ξ can be written in the form $x(u) = \xi + u + \varphi(u)v$, where $\varphi(u)$ is determined from the equation $y(\xi + u + \varphi(u)v) = 0$. It can be shown that

$$\varphi'(0)h = 0, \quad \varphi''(0) = \frac{P(\xi)y'(\xi)hk}{\|y'(\xi)\|^2} \quad (h, k \in U). \quad (8)$$

4. Nonhomogeneous equation. We proceed to the study of the nonhomogeneous equation (1) under the assumption that the conditions of complete solvability are satisfied.

The general solution of equation (1) has the form

$$y(x) = c^1 y_1(x) + \dots + c^{m+1} y_{m+1}(x) + \tilde{y}(x), \quad (9)$$

where $y_1(x), \dots, y_{m+1}(x)$ is a fundamental system of solutions of the corresponding homogeneous equation, and $\tilde{y}(x)$ is a particular solution of the nonhomogeneous equation.

A particular solution $y(x)$ of the nonhomogeneous equation with zero initial data has the form

$$y(x) = \int_{\xi}^x f(s)K(x, s) ds \quad (10)$$

(the Lagrange method (4)). Here the vector function

$$K(x, s) = \{K_1(x, s), \dots, K_m(x, s)\}, \quad (11)$$

which we shall call the Cauchy function, is determined as follows:

$$K_j(x, s) = \frac{\begin{vmatrix} y_1(s) & \dots & y_{m+1}(s) \\ \vdots & & \vdots \\ \frac{\partial y_1(s)}{\partial x^{j-1}} & \dots & \frac{\partial y_{m+1}(s)}{\partial x^{j-1}} \\ y_1(x) & \dots & y_{m+1}(x) \\ \frac{\partial y_1(s)}{\partial x^{j+1}} & \dots & \frac{\partial y_{m+1}(s)}{\partial x^{j+1}} \\ \vdots & & \vdots \end{vmatrix}}{\det W(s)} \quad (j = 1, \dots, m). \quad (12)$$

It follows from (12) that: 1) $K(s, s) = 0$; 2) $K(x, s)$ is continuous jointly in the variables x, s ; 3) $\partial K(s, s)/\partial x = 1$; 4) for fixed $s \in G$, each function $K_j(x, s)$ is a solution of the homogeneous equation $\mathcal{L}\{y\} = 0$. It can be shown that these properties uniquely determine the Cauchy function and, consequently, may be taken as its definition.

5. Multipoint problem. We now consider problem (1)–(3) under the assumption that $P(x) = Q(x) = 0$ and $\eta_i = 0$.

Theorem 4. Problem (1)–(3) for arbitrary $f(x)$ has a unique solution if and only if the points ξ_1, \dots, ξ_{m+1} lie in general position, i.e., are vertices of some m -dimensional simplex σ^m .

Let $\alpha^1(x), \dots, \alpha^{m+1}(x)$ be the barycentric coordinates and let the simplex σ^m lie in the domain G . Then the solution of problem (1)–(3) can be represented in the form

$$y(x) = - \sum_{i=1}^{m+1} \alpha^i(x) \int_{\xi_i}^x f(s)(s - \xi_i) ds \quad (13)$$

(in that part of the domain G which is star-shaped with respect to the simplex σ^m).

In the simplex σ^m the estimates

$$|y(x)| \leq \left(\sum_{i=1}^{m+1} \frac{\alpha^i(x) \|x - \xi_i\|^2}{2} \right) \max_{x \in \sigma^m} \|f(x)\| \leq \frac{1}{2} r^2 \max_{x \in \sigma^m} \|f(x)\|, \quad (14)$$

$$\|y'(x)\| \leq \left(\sum_{i=1}^{m+1} \frac{\|x - \xi_i\|^2}{2d_i} \right) \max_{x \in \sigma^m} \|f(x)\|. \quad (15)$$

hold.

The number r in inequality (14) has a simple geometric meaning: it is the radius of the ball circumscribed about the simplex σ^m (we do not give the analytic expression because of its cumbersome form). The numbers d_i in inequality (15) are the distances from the vertices ξ_i to the opposite faces of the simplex σ^m .

We note that in the case $m = 1$, inequalities (14) and (15) pass into the well-known estimates for the Green's function of the boundary-value problem.

Voronezh State
University

Received
6 VI 1964

REFERENCES CITED

1. F. und R. Nevanlinna, *Absolute Analysis*, Grundlehren Math. Wiss., Berlin, 1959, p. 102.
2. H. Hahti, T. Klemola, Ann. Acad. Sci. Fennicae, Ser. A, Math. I, 318, Helsinki, 1962.
3. T. Klemola, Ann. Acad. Sci. Fennicae, Ser. A, Math. I, 260, Helsinki, 1961.
4. I. G. Petrovsky, *Lectures on the Theory of Ordinary Differential Equations*, Moscow–Leningrad, 1952.
5. L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, Moscow, 1951.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.