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Abstract

Full Text

MATHEMATICS

E. V. VORONOVSKAYA

ODD POLYNOMIALS OF LEAST DEVIATION

(Presented by Academician S. N. Bernstein on 6 VI 1964)

In problems of the synthesis of antennas and electric circuits, one encounters the need to construct odd algebraic polynomials that deviate least from a given constant quantity on a finite segment. In this article analytic methods are proposed for constructing such polynomials by means of the method of functionals ⁽¹⁾.

§ 1. **Problem.** Among polynomials of the form

$$P_{2n+1}(x) = \sum_{k=0}^n p_k x^{2k+1} \quad (1)$$

find the one which on $[\lambda, 1]$ ($0 < \lambda < 1$) deviates least from A ($A > 0$).

The powers $(x^{2k+1})_0^n$ form a Chebyshev system of functions, since if the number of roots of (1) on $[\lambda, 1]$ is m , then the total number of roots on the axis (by oddness) is $2m + 1 \leq 2n + 1$, i.e. $m \leq n$.

Thus there exists $P_{2n+1}(x)$ —the polynomial of least deviation from A , moreover unique, and $A - P_{2n+1}(x) = Y_{2n+1}(x)$ has on $[\lambda, 1]$ no fewer than $n + 2$ points of deviation, at which it attains $\pm L$ in alternating order (Chebyshev alternance). Here $L = \max_{[\lambda, 1]} |Y_{2n+1}(x)|$ ⁽²⁾.

If s denotes the number of points of deviation, then it is easy to establish that $s = n + 2$, and that this number includes the endpoints λ and 1 .

Indeed, $Y'_{2n+1}(x)$ is an even polynomial, and the interior points of deviation, in number s_1 , are its roots. Then $2s_1 \leq 2n$; consequently, $s = n + 2$, with λ and 1 necessarily included in their number, not being roots of the derivative. Thus $Y'_{2n+1}(x)$ has at the endpoints the property of primitiveness.

It is obvious that if $Y_{2n+1}(x)$ has been found for the case $A = 1$ with deviation L , then for any other $A (> 0)$ the required polynomial will be $AY_{2n+1}(x)$ with deviation L . In what follows we take $A = 1$.

Let us note some consequences:

1. On $[-1, -\lambda]$, $P_{2n+1}(x)$ deviates least from -1 .

2. The passport $[2n + 1, n + 2, 0]$ belongs to class II, since

$$s = n + 2 > \frac{2n + 1}{2} + 1.$$

Consequently, the number of its independent parameters is $l = 2n + 2 - s = n$. Denote by $\lambda = \sigma_0 < \sigma_1 < \dots < \sigma_n < \sigma_{n+1} = 1$ all the points of deviation.

Transform $Y_{2n+1}(x)$ by the substitution $\bar{x} = (x - \lambda)/(1 - \lambda)$ and denote $Y_{2n+1}(\bar{x})/L = Q_{2n+1}(\bar{x})$. This latter belongs on $[0, 1]$ to the extremal polynomials of passport $[2n + 1, n + 2, 0]$ and is likewise primitive. The centers of symmetry of $Y_{2n+1}(x)$, $Y_{2n+1}(x)/L$, and $Q_{2n+1}(x)$ lie respectively at the points $(0, 1)$; $(0, 1/L)$; $(\gamma, 1/L)$, where $\gamma = -\lambda/(1 - \lambda) (< 0)$.

The stated problem has been reduced to finding, in the domain of polynomials of passport $[2n + 1, n + 2, 0]$, a primitive polynomial with center of symmetry at the prescribed point $x = \gamma$.

§ 2. The defining segment-functional of all polynomials (up to sign) of passport $[2n + 1, n + 2, 0]$ is as follows:

$$0_0, 0_1, \dots, 0_n, 1_{n+1}, \theta_1, \theta_2, \dots, \theta_n, \quad (2)$$

where $(\theta_i)_1^n$ are independent parameters of the family $\{Q_{2n+1}(x, \theta_1, \dots, \theta_n)\}$; they vary in the n -dimensional space M_n near the point $(\theta_1^*, \theta_2^*, \dots, \theta_n^*)$, obtained by the best continuation of the basis of the segment. At this point the extremal polynomial is

$$T_{n+1}(x) = \cos(n + 1) \arccos(2x - 1).$$

We denote by $\{\tilde{Q}_{2n+1}(x, \theta_1, \dots, \theta_n)\}$ the selection from this family consisting of all primitive polynomials and only of them:

$$\{\tilde{Q}_{2n+1}(x, \theta_1, \dots, \theta_n)\} \subset \{Q_{2n+1}(x, \theta_1, \dots, \theta_n)\}.$$

The fact that $\{\tilde{Q}_{2n+1}\}$ is only a part of $\{Q_{2n+1}\}$ follows from the fact that a polynomial of the form $T_{2n+1}(\alpha x + \beta)$ belongs to $\{Q_{2n+1}\}$, but not to $\{\tilde{Q}_{2n+1}\}$, if $\alpha x + \beta$ is a transformation chosen so that $T_{2n+1}(\alpha x + \beta)$ has exactly n nodes on $[0, 1]$. The fact that $\tilde{Q}_{2n+1}(x, \theta_1, \theta_2, \dots, \theta_n)$ admits an n -dimensional independent variation of its parameters near the interior nodes of $T_{n+1}(x)$ follows from the fact that $T_{2n+1}(x)$ itself is primitive.

Because in $\{\tilde{Q}_{2n+1}\}$ the nodes 0 and 1 are fixed, the variables are $(\sigma_i)_1^n$. This makes it possible not to resort to a complicated system of differential equations, but to solve the problem of constructing $\{\tilde{Q}_{2n+1}\}$ purely algebraically. For this

it is first necessary to establish the equivalence of the parameters $(\theta_i)_1^n$ to certain others.

Put

$$\prod_1^{n-1} (x - \sigma_i) = \alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1} + x^n.$$

We shall prove that

$$\{\tilde{Q}_{2n+1}(x, \theta_1, \dots, \theta_n)\} = \{\tilde{Q}_{2n+1}(x, \alpha_0, \dots, \alpha_{n-1})\} = \{\tilde{Q}_{2n+1}(x, \sigma_1, \dots, \sigma_n)\}, \quad (3)$$

where the groups $(\theta_i)_1^n$, $(\alpha_i)_0^{n-1}$, $(\sigma_i)_1^n$ are in one-to-one correspondence of their parameters in certain n -dimensional domains.

Let

$$R_{n+2}(x) = \prod_1^n (x - \sigma_i) x(x - 1) = \beta_1 x + \beta_2 x^2 + \dots + \beta_{n+1} x^{n+1} + x^{n+2}$$

be the full resolvent of \tilde{Q}_{2n+1} . According to its basic property, from (2) we have:

$$\beta_{n+1} + \theta_1 = 0, \quad \beta_n + \beta_{n+1} \theta_1 + \theta_2 = 0, \dots, \quad \beta_2 + \beta_3 \theta_1 + \dots + \beta_{n+1} \theta_{n-1} + \theta_n = 0,$$

i.e. $(\beta_i)_2^{n+1}$ and $(\theta_i)_1^n$ determine one another uniquely. Further,

$$\alpha_{n-1} - 1 = \beta_{n-1}; \quad \alpha_{n-2} - \alpha_{n-1} = \beta_n; \quad \dots, \quad \alpha_0 - \alpha_1 = \beta_2$$

and

$$\beta_1 = -\alpha_0.$$

But the coefficients (α_i) and the roots (σ_i) of the polynomial

$$\prod_1^n (x - \sigma_i)$$

determine one another uniquely. Consequently, the assertions (3) are proved.

It is most convenient to choose the form $\tilde{Q}_{2n+1}(x, \sigma_1, \sigma_2, \dots, \sigma_n)$, where $(\sigma_i)_1^n$ are independent and vary correspondingly near the interior nodes of $T_{n+1}(x)$.

§ 3. We proceed to methods of constructing $\tilde{Q}_{2n+1}(x, \sigma_1, \dots, \sigma_n)$.

1a) n even. The distribution of the polynomial is as follows:

$$0 < \sigma_1^+ < \dots < \sigma_n^- < 1.$$

We have the basic identity for polynomials of class II:

$$x \prod_1^{n/2} (x - \sigma_i^-)^2 \psi_n(x) + (1-x) \prod_1^{n/2} (x - \sigma_i^+)^2 \varphi_n(x) \equiv 2,$$

where the unknown polynomials $\psi_n(x)$ and $\varphi_n(x)$ are found uniquely by the method of successive division. Then

$$\begin{aligned} \tilde{Q}_{2n+1}(x, \sigma_1, \dots, \sigma_n) &= 1 - (1-x) \prod_1^{n/2} (x - \sigma_i^+)^2 \varphi_n(x) = \\ &= -1 + x \prod_1^{n/2} (x - \sigma_i^-)^2 \psi_n(x). \end{aligned}$$

b) n is odd. The distribution is as follows: $0_1^+ < \sigma_1^- < \dots < \sigma_n^- < 1^+$. Then

$$x(1-x) \prod_1^{(n-1)/2} (x - \sigma_i^+)^2 \varphi_n(x) + \prod_1^{(n+1)/2} (x - \sigma_i^-)^2 \psi_n(x) \equiv 2,$$

$$\begin{aligned} \tilde{Q}_{2n+1}(x, \sigma_1, \dots, \sigma_n) &= 1 - x(1-x) \prod_1^{(n-1)/2} (x - \sigma_i^+)^2 \varphi_n(x) = \\ &= -1 + \prod_1^{(n+1)/2} (x - \sigma_i^-)^2 \psi_n(x). \end{aligned}$$

It remains now, from the polynomials found,

$$\tilde{Q}_{2n+1}(x, \sigma_1, \dots, \sigma_n) = \tilde{q}_{2n+1}(\sigma_1, \dots, \sigma_n)x^{n+1} + \dots + q_0(\sigma_1, \dots, \sigma_n) \quad (4)$$

to make a selection $\{\tilde{Q}_{2n+1}^{(0)}(x, \gamma)\}$ of polynomials with center of symmetry $x = \gamma$ ($\gamma < 0$ arbitrary). $\tilde{Q}_{2n+1}^{(0)}$ is a one-parameter family, since, according to § 1, the choice of γ completely determines the polynomial—the solution of the Chebyshev problem posed.

Making the substitution $x = t + \gamma$, we have

$$Q_{2n+1}^{(0)}(t + \gamma, \sigma_1, \dots, \sigma_n) = \sum_{k=1}^n \varphi_k(\gamma, \sigma_1, \dots, \sigma_n) t^{2k+1},$$

where the coefficients of t^2, t^4, \dots, t^{2n} are equal to zero; from these n equations the $\sigma_k = \sigma_k(\gamma)$ are determined, moreover uniquely.

2. Let us give another method, which at once yields the mentioned one-parameter selection $Q_{2n+1}^{(0)}(x, \gamma)$. The polynomial $\tilde{Q}_{2n+1}^{(0)'}(x, \gamma)$ has roots $(\sigma_i)_1^n$ and $(2\gamma - \sigma_i)_1^n$. Therefore,

$$\tilde{Q}_{2n+1}^{(0)'}(x, \gamma) = A \prod_1^n (x - \sigma_i) \prod_1^n (x - 2\gamma + \sigma_i),$$

where $A = \text{const}(x)$,

$$\tilde{Q}_{2n+1}^{(0)}(x, \gamma) = A \int_0^x \prod_1^n (x - \sigma_i) \prod_1^n (x - 2\gamma + \sigma_i) dx + B.$$

Denote the integral by $I(0; x)$ and write the boundary conditions:

$$Q_{2n+1}^{(0)}(0, \gamma) = (-1)^n, \quad \tilde{Q}_{2n+1}^{(0)}(1, \gamma) = 1, \quad \tilde{Q}_{2n+1}^{(0)}(\sigma_i, \gamma) = (-1)^{n-1+i}.$$

Then $B = (-1)^{n-1}$, $A = \frac{2}{I(0, 1)}$ for n even; for n odd $I(0, 1) = 0$. Further, $I(\sigma_i, \sigma_{i+2}) = 0$ for $i = 0, 1, \dots, n-1$ ($\sigma_0 = 0$; $\sigma_{n+1} = 1$), i.e.

$$\tilde{Q}_{2n+1}^{(0)}(\sigma_i, \sigma_{i+1}) = -\tilde{Q}_{2n+1}^{(0)}(\sigma_{i+1}, \sigma_{i+2}).$$

Geometrically this means that if one draws the curve

$$\prod_1^n (x - \sigma_i) \times \\ \times \prod_1^n (x - 2\gamma - \sigma_i),$$

then all the areas on the intervals $(0, \sigma_1), (\sigma_1, \sigma_2), \dots, (\sigma_n, 1)$ are equal to one another in absolute value. Denote this quantity by S_0 ; then $A = 2/S_0$.

Finally, it is easy to verify that for any $\gamma (< 0)$ the polynomial found

$$\widetilde{Q}_{2n+1}^{(0)}(x, \gamma) = \frac{2}{S_0} I(0, x) + (-1)^{n-1}$$

is reduced on $[0, 1]$. Indeed, for even n we have $0 \leq I(x) \leq S_0$ for $0 \leq x \leq 1$; for odd n we have $-S_0 \leq I(x) \leq 0$ (which is evident geometrically), and then in both cases

$$-1 \leq \frac{2}{S_0} I(x) + (-1)^{n-1} \leq 1 \quad \text{for } 0 \leq x \leq 1.$$

Thus, the second method shows that for a given γ the unknowns $(\sigma_i)_1^n$ should be sought respectively near

$$\sin^2 \frac{i\pi}{2(n+1)} \quad (i = 1, 2, \dots, n)$$

so that the indicated areas are equal in magnitude.

In conclusion, we note that equality of the areas between nodes also holds for other cases of extremal polynomials, for example for the Zolotarev polynomials, which form a one-parameter family of primitive polynomials

$$Z_n(x, \vartheta) = \vartheta x^n + \varphi_1(\vartheta)x^{n-1} + \dots + \varphi_{n-1}(\vartheta)x + (-1)^n$$

of passport $[n, n, 0]$ (1). Here

$$Z'_n(x, \vartheta) = n\vartheta \prod_2^{n-1} (x - \sigma_i)(x - \lambda),$$

where $\sigma_i = \sigma_i(\vartheta)$ and $\lambda = \lambda(\vartheta)$ is the root of the derivative that traverses $(1, +\infty)$ as

$$2^{2n-3} < \vartheta < \cos^2 \frac{2\pi}{2n} 2^{2n-1}.$$

Then

$$Z_n(x, \vartheta) = n\vartheta \int_0^x \prod_1^{n-1} (x - \sigma_i)(x - \lambda) dx + (-1)^n = n\vartheta I(0, x) + (-1)^n$$

with analogous boundary conditions:

$$Z_n(1, \vartheta) = -1; \quad Z_n(\sigma_i, \vartheta) = (-1)^{n+i} \quad (i = 2, 3, \dots, n).$$

Here $S_0 = 2/n\vartheta$.

Thus, by prescribing the value ϑ from the indicated interval, we obtain a concrete Zolotarev polynomial, choosing $(\sigma_i)_1^{n-1}$ near the interior nodes of $T_{n-1}(x)$ and $\lambda > 1$ so that the equality $|I(\sigma_i, \sigma_{i+1})| = 2/n\vartheta$ is satisfied. Here such a method is apparently practically inconvenient, since the interval for λ is unbounded.

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CITED LITERATURE

- ¹ E. V. Voronovskaya, *The Method of Functionals and Its Applications*, 1963.
- ² S. N. Bernstein, *Extremal Properties of Polynomials*, 1937.

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