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Abstract

Full Text

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CARDINALITIES OF A BASE OF AN H-CLOSED SPACE

(Presented by Academician P. S. Aleksandrov on 11 VII 1963)

In this note I shall prove

Theorem 1. Let X be an H-closed space and let B be its t-pseudobase* such that, for every point $x \in X$, the set B_x of all elements of the t-pseudobase B for which the point x is a point of contact has cardinality $\leq a$, where a is some cardinal number. Then the cardinality of B is $\leq a$.

In addition to this theorem, here a theorem on the cardinality of t-pseudobases in bicompact spaces is proved, proved by A. Mishchenko for bases (^{1}). In conclusion an example is given of an H-closed space showing that Theorem 1 cannot be strengthened.

Proof of Theorem 1. I carry out the proof following (^{1}). I shall call a **pseudocovering** of a space X any finite system $\omega = \{U_\alpha\}$ of open sets such that the system $\bar{\omega} = \{[\overline{U_\alpha}]\}$, composed of the closures of the elements of the system ω, covers the space X. As is known (^{2}), from any covering of an H-closed space one can choose a finite pseudocovering. I shall call a pseudocovering ω **minimal** if it is finite and such that for every $U_\alpha \in \omega$ there is a point x not belonging to any other elements of the pseudocovering ω. Let V be the set of possible minimal pseudocoverings of the H-closed space X, composed of elements of the t-pseudobase B.

Since each minimal pseudocovering is finite, the cardinality of the set V does not exceed the cardinality of B. But the cardinality of the set V cannot be less than the cardinality of B; this follows from the fact that for any $\sigma \in B$ there exists a pseudocovering $\omega \in V$ such that $\sigma \in \omega$.

To prove the last assertion, take some point $x \in \sigma$. For each point $y \in X \setminus x$ choose an element $U(y)$ of the t-pseudobase B containing it and lying in $X \setminus x$. Denote by Ω the covering consisting of all the just selected $U(y)$ and of σ. Let ω be any minimal pseudocovering contained in Ω; if one adds σ to its elements (in the case when σ is not already in ω), then again a minimal pseudocovering is obtained. Thus, the cardinality of the set V is equal to the cardinality of B. Suppose that the cardinality of $B > a$. Then, if V_n denotes the set of all minimal pseudocoverings consisting of n distinct elements of the t-pseudobase, there exists such an n_0 that the cardinality of $V_{n_0} > a$.

Let $k \leq n_0$, and let $\sigma_1, \dots, \sigma_k$ be arbitrary distinct elements of the t-pseudobase B. Denote by $S_{\sigma_1 \dots \sigma_k}$ the set of all those pseudocoverings $\omega \in V_{n_0}$ for which

$\sigma_i \in \omega$ $(i = 1, 2, \dots, k)$.

Let x be an arbitrary point of the space X, and let B_x be the set of all elements of the t-pseudobase B for which the point x serves as a point of contact. Then the equality holds

$$V_{n_0} = \bigcup_{\sigma \in B_x} S_\sigma. \tag{1}$$

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* A t-pseudobase of a topological space X is a system of open sets such that for every point $x \in X$ there is in it a subsystem whose intersection is exactly the point x. A base, obviously, is a t-pseudobase.

If $\sigma_1, \dots, \sigma_k$ are any (pairwise distinct) elements of the t-pseudobase B not containing the point x, then

$$S_{\sigma_1 \dots \sigma_k} = \bigcup_{\sigma \in B_x} S_{\sigma_1 \dots \sigma_k \sigma}, \quad \text{where } \sigma \neq \sigma_i \quad (i = 1, \dots, k). \tag{2}$$

Let $x_1 \in X$ be an arbitrary point. There exists $\sigma_1 \in B_{x_1}$ such that the cardinality of S_{σ_1} is $> a$ (this follows from equality (1)). Suppose that for $k < n_0$ such (pairwise distinct) elements $\sigma_1, \dots, \sigma_k$ of the t-pseudobase B have been found that the cardinality of $S_{\sigma_1 \dots \sigma_k}$ is $> a$. Since $k < n_0$, there exists a point x_{k+1} that is contained in none of the sets $\sigma_1, \dots, \sigma_k$. But then, by virtue of equality (2), there exists also such a $\sigma_{k+1} \in B_{x_{k+1}}$ that the cardinality of $S_{\sigma_1 \dots \sigma_k \sigma_{k+1}}$ is $> a$.

As a result, for all $k \leq n_0$, the sets $S_{\sigma_1 \dots \sigma_k}$ have cardinality greater than a. In particular, the cardinality of the set $S_{\sigma_1 \dots \sigma_{n_0}}$ exceeds the cardinal number a. But $S_{\sigma_1 \dots \sigma_{n_0}} \subset V_{n_0}$, and therefore there exists only one cover $\omega \in S_{\sigma_1 \dots \sigma_{n_0}}$, namely $\omega = \{\sigma_1, \sigma_2, \dots, \sigma_{n_0}\}$. The theorem is proved.

Theorem 2. *Let X be a bicompact T_1-space and let B be its t-pseudobase such that, for every point $x \in X$, the set B_x of all elements of the t-pseudobase B containing the point x has cardinality $\leq a$, where a is some cardinal number. Then the cardinality of B is $\leq a$.*

The proof of this theorem differs hardly at all from the proof of the corresponding theorem in paper (^{1}).

Now I shall construct an H-closed space M possessing a point-countable base but not possessing a countable base. This will show that Theorem 1 cannot be strengthened. The space is borrowed from paper (^{1}).

Represent the interval $[0, 1]$ as the sum of a disjoint system N_1 of nonempty everywhere dense sets M_α:

$$[0, 1] = \bigcup_{\alpha < \omega_1} M_\alpha, \quad M_\alpha \cap M_\beta = \Lambda, \quad \alpha \neq \beta.$$

Let p_1, p_2 be arbitrary rational numbers, $p_1 < p_2$. By (p_1, p_2) we denote the interval of the number line with endpoints p_1 and p_2. Let x be an arbitrary point of the interval $[0, 1]$, $x \in M_\alpha$. Define the neighborhood $U_{p_1p_2}(x)$ as the set

$$U_{p_1p_2}(x) = (p_1, p_2) \cap \bigcup_{\beta \geq \alpha} M_\beta.$$

In (^{1}) it is proved that M is a Hausdorff space without a countable, but with a point-countable, base. I shall prove that M is H-closed.

Take an arbitrary cover ω of the space M by elements of the base. We shall prove that from the cover ω one can extract a finite pseudocover π. Let $U_{p_1p_2}(x) \in \omega$; then the interval $(p_1, p_2) \in \overline{\omega}$; thus we obtain a cover $\overline{\omega}$.

From the cover $\overline{\omega}$ extract a finite cover $\overline{\pi}$ of the interval $[0, 1]$. Let $(p_1, p_2) \in \overline{\pi}$; then the corresponding $U_{p_1p_2}(x) \in \pi$; thus we obtain π. It is not hard to verify that the closure of each $U_{p_1p_2}(x)$ is exactly the interval $[p_1, p_2]$. Hence it follows that π is a finite pseudocover.

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Note: Figure translations are in progress. See original paper for figures.

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