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Abstract

Full Text

MATHEMATICS

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ON POSITIONAL OPERATIVES

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In the present note we study algebras with one n -ary operation subject to certain conditions of associativity type.

1. A set S with one everywhere-defined n -ary operation is called an n -**operative**. We shall write this operation in the form

$$(x_1 x_2 \dots x_n) = x \quad (x_i, x \in S).$$

Let

$$((x_1 x_2 \dots x_n) x_{n+1} \dots x_{2n-1}) = (x_1 x_2 \dots x_{2n-1}).$$

If A_i are nonempty subsets of the n -operative S , then by $(A_1 A_2 \dots A_n)$ we denote the set of all products $(a_1 a_2 \dots a_n)$, where $a_i \in A_i$; if, in addition, one of the sets A_i (for example, A_1) consists of a single element a , then we shall write $(A_1 A_2 \dots A_n) = (a A_2 \dots A_n)$. If $A_{t+1} = A_{t+2} = \dots = A_{t+k} = A$, then we denote

$$(A_1 A_2 \dots A_n) = (A_1 A_2 \dots A_t A^k A_{t+k+1} \dots A_n);$$

A^0 is the empty symbol.

Let S be an n -operative, and let φ be a permutation of the indices $1, 2, \dots, n$. The set S , with respect to the n -ary operation

$$[x_1 x_2 \dots x_n] = (x_{\varphi 1} x_{\varphi 2} \dots x_{\varphi n}),$$

is also an n -operative, which we denote by S_φ . If $\varphi 1 = 1$, $\varphi n = n$, then the operative S_φ will be called **similar** to the operative S .

2. Let $\rho = \{\rho_k\}_{k=1}^n$ be a system of (one-to-one) permutations of the indices $1, 2, \dots, 2n - 1$ such that $\rho_1 = \rho_n = \varepsilon$ (ε is the identity permutation). An n -operative S is called an $n\rho$ -**operative** if for any k ($1 \leq k \leq n$) and all $x_i \in S$,

$$(x_1 x_2 \dots x_{k-1} (x_k x_{k+1} \dots x_{k+n-1}) x_{k+n} \dots x_{2n-1}) = (x_{\rho_k 1} x_{\rho_k 2} \dots x_{\rho_k (2n-1)}).$$

Let $\pi = \{\sigma_k, \pi_k\}_{k=1}^n$ be a system of pairs of permutations of the n symbols $1, 2, \dots, n$ such that $\sigma_1 = \sigma_n = \pi_1 = \pi_n = \varepsilon$, $\sigma_k k = k$ for all k . An n -operative S is called an $n\pi$ -**operative** if for all $x_i, y_j \in S$ and any k ($1 \leq k \leq n$),

$$(x_1 x_2 \dots x_{k-1} (y_1 y_2 \dots y_n) x_{k+1} \dots x_n) = (x_{\sigma_k 1} x_{\sigma_k 2} \dots x_{\sigma_k (k-1)} y_{\pi_k 1} \dots \\ \dots y_{\pi_k n} x_{\sigma_k (k+1)} \dots x_{\sigma_k n}).$$

An $n\pi$ -operative will be called: an n -**associative**, if $\sigma_k = \pi_k = \varepsilon$ for $k = 1, 2, \dots, n$; an n -**alternative**, if n is odd and all $\sigma_k = \pi_{2j+1} = \varepsilon$, $\pi_{2j} = \tau$, where $\tau i = n - i + 1$ for $i = 1, 2, \dots, n$; an $n\pi$ -**commutative**, if $\sigma_k = \varepsilon$ for every k ($1 \leq k \leq n$) and

$$(x_1 x_2 \dots x_{n-1} x_n) = (x_n x_2 \dots x_{n-1} x_1)$$

for any $x_i \in S$. For $n = 2$ there exists a unique $n\rho$ -operative, which is a 2-associative, i.e. an ordinary semigroup ⁽¹⁾. A 3-alternative is V. V. Wagner' s semigroup ^(2,3). Some results on n -associatives (for arbitrary n) can be found in notes ⁽⁴⁻⁶⁾.*

3. An element a of an n -operative S is called (j) -**invertible** if

$$(S^{j-1} a S^{n-j}) = S.$$

An element a that is simultaneously (1)-invertible and (n) -invertible is called **two-sided invertible**.

* The terms "operative" and "associative" were taken by me from works ^(7,8). In ⁽⁸⁾ only 2-operatives are considered.

One can give an example of an n -associative S containing an element a that is (k) -invertible for $k = 1, 2, \dots, n - 1$, but is not (n) -invertible.

Theorem 1. Every two-sided invertible element of a pl-operative S is (k) -invertible for every k .

Theorem 2. The set of all two-sided invertible elements of a pl-operative S is its suboperative.

Theorem 3. If a pr-operative S contains a suboperative T , all elements of which are two-sided invertible in S , then S is similar to some pl-operative S_φ .

By the definition of similarity (see item 1), in this case all elements of T_φ are two-sided invertible in S_φ .

4. Let A be a finite group of order k , $n = kt + 1$, where t is an integer > 0 ; let ν be a one-to-one mapping of the set $\{1, 2, \dots, n\}$ onto A . Instead of $\nu(i) = a \in A$ we shall write $a = a_i$. We shall call ν a **numbering** of the group A if $a_1 = e$ (e is the identity of the group A) and from $i \equiv j \pmod{k}$ it follows that $a_i = a_j$. The group A with numbering ν will be denoted by A_ν . The cyclic group generated by an element a of order k , with numbering $a_i = a^{i-1}$ ($i = 1, 2, \dots, n$), will be denoted by $[a]$.

To each element $a_j \in A_\nu$ there corresponds a permutation τ_j of the set $\{1, 2, \dots, n-1\}$ such that: a) $a_{\tau_j i} = a_j a_i$; b) if $ks < i \leq k(s+1)$, then also $ks < \tau_j i \leq k(s+1)$. Let π_j ($j = 1, 2, \dots, n$) be the following permutation of the set $\{1, 2, \dots, n\}$: $\pi_j i = \tau_j^{-1}(i+j-1)$, if $i \leq n-j$, $\pi_j i = \tau_j^{-1}(i+j-n)$, if $n-j < i < n$, $\pi_j n = n$. The system of pairs of permutations $\{\sigma_j, \pi_j\}_{j=1}^n$, where all $\sigma_j = \varepsilon$, will be denoted by π_{A_ν} .

Theorem 4. If a pl-operative S contains at least one two-sided invertible element, then S is either an n -associative, or an n -alternative, or a pl-commutative operative.

5. Let D be a semigroup, whose operation we shall denote by $*$; let a be an arbitrary two-sided invertible element of D ; let Ψ be a finite subset of the extended automorphism group of the semigroup D (i.e., the elements ψ_i of the set Ψ are automorphisms or inverse automorphisms of the semigroup D); $\psi_i a = a$ for all $\psi_i \in \Psi$; it is possible that $\psi_i = \psi_j$ for $i \neq j$. Introduce on the set D an n -ary operation: for arbitrary $x_i \in D$

$$(x_1 x_2 \dots x_n) = x_1 * \psi_2 x_2 * \psi_3 x_3 * \dots * \psi_{n-1} x_{n-1} * a * x_n.$$

The n -operative obtained in this way will be denoted by $S = S(D, a, \Psi, n)$. If Ψ is a group and the indices of ψ_i correspond to some numbering ν of it (see item 4), then we denote $S = S(D, a, \Psi_\nu, n)$.

Let, further, λ_a be the inner automorphism of the semigroup D generated by the element a : $\lambda_a x = a^{-1} * x * a$. If ψ is an automorphism, $\psi a = a$, $\psi^{n-1} = \lambda_a^{-1}$, $\psi_i = \psi^{i-1}$ ($i = 1, 2, \dots, n-1$), then we denote $S = S(D, a, \psi, n)$. If $n = 2m+1$, ψ is an automorphism, ξ is an involution, $\psi^m = \lambda_a^{-1}$, $\xi \psi = \lambda_a \psi^{m-1} \xi$, $\psi a = \xi a = a$, $\psi_{2j+1} = \psi^j$, $\psi_{2j} = \psi^{j-1} \xi$, then we denote $S = S(D, a, \psi, \xi, n)$. $S(D, a, \psi, n)$ is n -associative, $S(D, a, \psi, \xi, n)$ is n -alternative; if D is commutative, then $S(D, a, \Psi_\nu, n)$ is pl_{Ψ_ν} -commutative (see items 2, 4).

6. **Theorem 5.** If an n -associative S contains at least one two-sided invertible element, then it is isomorphic to some operative $S(D, a, \psi, n)$, where ψ is an automorphism of the semigroup D and $\psi a = a$.

Theorem 6. If an n -alternative S contains at least one two-sided invertible element, then it is isomorphic to some operative $S(D, a, \psi, \xi, n)$, where ψ is an automorphism of the semigroup D , ξ is its involution, $\psi a = \xi a = a$.

Theorem 7. If a pl-commutative S contains at least one (1)-invertible element, then it is similar to the operative $S(D, a, \Psi, n)$ (see items 4, 5).

where D is a commutative semigroup, Ψ is a subgroup of the group of its automorphisms, $\psi_i a = a$ for any $\psi_i \in \Psi$.

This theorem is very close to Evans' results ⁽⁹⁾.

7. A subset A of an n -operative S is called an (r) -ideal ⁽¹⁰⁾ if $(S^{r-1}AS^{n-r}) \subseteq A$. A (1)-ideal is called a right ideal, and an (n) -ideal a left ideal. If an np -operative S contains no proper left or right ideals, then every element of it is two-sided invertible.

From Theorems 3-7 it follows:

Theorem 8. Every np -operative (nl -operative) without proper left and right ideals is similar (respectively isomorphic) to some operative

$$S = S(D, a, \Psi, n),$$

where D is a group; moreover S satisfies the conditions of one of Theorems 5-7. In particular, every n -associative without proper left and right ideals is isomorphic to some operative

$$S(D, a, \psi, n),$$

where D is a group and ψ is its automorphism.

An n -associative S is called an n -Dörnte group ^(11,12) if

$$(x_1 x_2 \dots x_{k-1} S x_{k+1} \dots x_n) = S$$

for any k ($1 \leq k \leq n$) and any $x_i \in S$.

Theorem 9. Every n -Dörnte group is isomorphic to some operative

$$S(D, a, \psi, n),$$

where D is a group, ψ is its automorphism, $\psi a = a$.

8. An element e of an n -operative S is called unitary (in the terminology of B. V. Wagner ⁽²⁾, biunitary) if

$$(x e^{n-1}) = (e^{n-1} x) = x$$

for any $x \in S$.

From Theorems 1-7 it follows:

Theorem 10. If an np -operative (nl -operative) contains a unitary element, then it is similar (respectively isomorphic) to an operative

$$S(D, e, \Psi, n),$$

where D is a semigroup with identity e , and the set Ψ satisfies the conditions of one of Theorems 5-7.

Theorem 11. An np -operative S can be embedded in an np -operative \bar{S} with unitary element S if and only if S is similar to an n -associative, an n -alternative, or to some nl -commutative.

Theorem 12. An nl -operative S can be embedded in an nl -operative \bar{S} with unitary element e if and only if S is n -associative or n -alternative, or is similar to some nl_{Ψ} -commutative.

Theorem 13. Every n -alternative S can be embedded in a semigroup (see item 2).

An element e of an operative S is called an identity if

$$(e^{k-1}xe^{n-k}) = x$$

for any $x \in S$ and every k .

From Theorem 8 follows the following theorem:

Theorem 14. If an np -operative (nl -operative) S contains an identity, then it is similar (respectively isomorphic) to the operative

$$S(D, e, \varepsilon, n)$$

(see item 5), where D is a semigroup with identity e , and ε is its identity automorphism.

The structure of an n -associative with identity was studied in the note ⁽⁶⁾.

Theorem 15. An np -operative (nl -operative) S can be embedded in an np -operative \bar{S} with identity e if and only if S is similar to an n -associative (respectively is an n -associative).

An np -operative S for $n > 2$ may contain several unitary elements or several identities. The set of all identities of an n -associative S (as also the set of all biunitary elements of a semigroup—see ⁽²⁾) is a suboperative of S .

Theorems 11, 12, and 15 can be somewhat strengthened, namely: the operative \bar{S} can be chosen so that the element e is its unique unitary element.

9. Somewhat aside from the preceding results are the following two theorems.

Theorem 16. Let A be a (k) -ideal of an $n\ell$ -operative S , $k \neq 1$, $k \neq n$. If

$$(A^n) = A,$$

then A is a (j) -ideal of the operative S for any j .

Theorem 17. Let A be simultaneously a left and a right ideal of an $n\ell$ -operative S . If $(A^n) = A$, then A is a (k) -ideal of the operative S for every k .

10. In connection with the results of Sections 4, 6-8, the question arises of isomorphism of operatives $S(D, a, \Psi, n)$ in the similarity of $n\pi$ -operatives.

Let D and D' be semigroups with identities e and e' , and let a and a' be their invertible elements; Ψ and Ψ' are subsets of their extended automorphism groups; c is an invertible element of D' ; f is an isomorphism of the semigroup D onto D' . Denote $c_1 = e'$, $c_k = \psi'_2 c * \psi'_3 c * \dots * \psi'_k c$ ($k = 2, 3, \dots, n-1$, $\psi'_i \in \Psi'$), $\xi_k x = c_k^{-1} * x * c_k$ ($x \in D'$). Suppose that f and c satisfy the conditions: $fa = c_{n-1} * a' * c$ and $\psi'_k = \xi_{k-1} f \psi_k f^{-1}$, if ψ_k is an inverse automorphism ($\psi_k \in \Psi$); $\psi'_k = \xi_k f \psi_k f^{-1}$, if ψ_k is an automorphism. Then the mapping $\varphi(x) = c * f(x)$ is an isomorphism of the operative $S(D, a, \Psi, n)$ onto $S(D', a', \Psi', n)$. We shall say that φ is generated by the isomorphism f and the element $c \in D'$.

Theorem 18. Every isomorphism of the operative $S(D, a, \Psi, n)$ onto $S(D', a', \Psi', n)$ is generated by some isomorphism f of the semigroup D onto D' and an element $c \in D'$.

11. Let S be an arbitrary n -operative. The operative S' , defined on the set S with the operation

$$[x_1 x_2 \dots x_n] = (x_n x_{n-1} \dots x_2 x_1) \quad (x_i \in S)$$

will be called the **inverse** ⁽²⁾ with respect to the operative S .

Theorem 19. Let S be an arbitrary $n\pi$ -operative containing a two-sided invertible element and not being $n\pi$ -commutative. If the operative S_φ (see Section 1) is an $n\pi$ -operative, then either $S_\varphi = S$, or $S_\varphi = S'$, where S' is the operative inverse with respect to S . In particular, if the $n\pi$ -operative S_φ is similar to the operative S , then $S_\varphi = S$.

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Note: Figure translations are in progress. See original paper for figures.

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