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Abstract

Full Text

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ON MULTIPLICATIVE UNIFORMLY CONTINUOUS FAMILIES OF RANDOM PROBABILITY OPERATORS

(Presented by Academician A. N. Kolmogorov on 27 III 1964)

1. Consider a set Y with a distinguished σ-algebra $\mathfrak{B}$ of subsets on it, containing all one-point subsets of Y. Denote by B the linear normed space of numerical functions $f(y)$, defined on Y and $\mathfrak{B}$-measurable, with norm

   $$\|f\| = \sup_{y \in Y} |f(y)|.$$

   A linear operator Π, defined on B, is called a **probability** operator if it takes nonnegative functions into nonnegative functions and $\Pi 1 = 1$.

Let $\{\Omega, \mathfrak{F}\}$ be a measurable space ($\mathfrak{F}$ is a σ-algebra of subsets of Ω). Consider a family of σ-algebras $\mathfrak{F}_t^s$, defined for $0 \leq s \leq t$ and satisfying the condition: if $[s_1, t_1] \subset [s, t]$, then

$$\mathfrak{F}_{t_1}^{s_1} \subset \mathfrak{F}_t^s \subset \mathfrak{F}.$$

Suppose that for all $0 \leq s < t$ and $\omega \in \Omega$ probability operators $\Pi_t^s(\omega) = \Pi_t^s$ on B are defined, satisfying the conditions:

M-1. For all $f \in B$ and $s < t$, the quantity $\Pi_t^s f(y)$, as a function of ω and y, is measurable with respect to the σ-algebra $\mathfrak{F}_t^s \times \mathfrak{B}$.

M-2. For $s < t < u$ the relation

$$\Pi_t^s \Pi_u^t = \Pi_u^s$$

holds.

Such a family of operators will be called a **multiplicative family of probability operators**.

Suppose also that condition M-3 holds. For all $\omega \in \Omega$ and $T > 0$,

$$\lim_{h \to 0} \sup_{0 \leq s \leq T} \sup_{0 < t - s < h} \|\Pi_t^s - I\| = 0,$$

where I is the identity operator.

Then the family is called **uniformly continuous**. A family of operators satisfying conditions M-1, M-2, M-3 will be called an M**-family**.

In addition to multiplicative families of operators, certain additive families of operators, called A**-families**, will be used; their definition is given below.

A family of linear operators $A_t^s(\omega) = A_t^s$ on B, defined for all $0 \le s < t$ and $\omega \in \Omega$, is called an A**-family** if the following conditions are satisfied:

A-1. Whatever $0 \le s < t$ and $\omega \in \Omega$ may be, the operator $I + \varepsilon A_t^s$ is a probability operator for $\varepsilon < \|A_t^s\|^{-1}$.

A-2. The function $A_t^s(\omega) f(y)$ is measurable with respect to $\mathfrak{F}_t^s \times \mathfrak{B}$, for fixed s and t, for all f from B.

A-3. For all $\omega \in \Omega$ and $T > 0$,

$$\lim_{h\to 0} \sup_{0\le s\le T} \sup_{0<t-s<h} \|A_t^s\| = 0.$$

There exists a one-to-one correspondence between M-families and A-families.

Theorem. To every M-family Π_t^s there corresponds an A-family A_t^s, connected with Π_t^s by the equation: for $s \in [0,t]$

$$\Pi_t^s = I + \int_s^t (d_u A_t^u)\Pi_t^u. \tag{1}$$

To every A-family A_t^s, equation (1) assigns a unique M-family Π_t^s, which is expressed in terms of A_t^s by the formula

$$\Pi_t^s = I + \int_s^t d_u A_t^u + \cdots + \int_{s\le u_1<\cdots<u_k\le t} d_{u_1} A_t^{u_1} \cdots d_{u_k} A_t^{u_k} + \cdots. \tag{2}$$

Thus, the description of M-families can be reduced to the description of the corresponding A-families, which in some cases is much simpler.

Remark 1. Every linear operator C on B has the form

$$Cf(y) = \int f(y')\, c(y, dy'),$$

where $c(y,A)$, for fixed y, is a countably additive function of bounded variation on $\mathfrak{B}$, and for fixed $A \in \mathfrak{B}$ the function $c(y,A)$ belongs to B. Let

$$\Pi_t^s f(y) = \int \pi_t^s(y, dy')\, f(y'), \qquad A_t^s f(y) = \int \alpha_t^s(y, dy')\, f(y'),$$

and let $\delta(y,A) = 1$ for $y \in A$, $\delta(y,A) = 0$ for $y \notin A$. Then relations (1) and (2) may be rewritten in the form

$$\pi_t^s(y,A) = \delta(y,A) + \int_s^t \int_Y \pi_t^u(y', A)\, d_u \alpha_t^u(y, dy'), \tag{3}$$

$$\pi_t^s(y, A) = \delta(y, A) + \cdots$$

$$\cdots + \int_{s \le u_1 < \cdots < u_k \le t} \int_Y \cdots \int_Y d_{u_1} \alpha_t^{u_1}(y, dy_1) \cdots d_{u_k} \alpha_t^{u_k}(y_{k-1}, A) + \cdots . \tag{4}$$

In the proof of the theorem it is established that $d_s \pi_t^s(y, dy')$ and $d_s \alpha_t^s(y, dy')$ have bounded variation on $[0, T] \times Y$, so that the integrals in (3) and (4) have the usual meaning.

Remark 2. If a probability measure P is given on Ω and the relations A-2, A-3, M-2, M-3 hold with probability 1, then the theorem remains valid if one requires (1) and (2) also to hold only with probability 1.

Remark 3. If instead of conditions M-3 and A-3 one requires the fulfillment of the conditions:

M-4. For every $\varepsilon > 0$, with probability 1,

$$\lim_{h \to 0} \sup_{0 \le s \le T} \sup_{0 < t - s < h} P\{\|\Pi_t^s - Y\| > \varepsilon / \mathfrak{F}_s^0\} = 0.$$

A-4. For every $\varepsilon > 0$, with probability 1,

$$\lim_{h \to 0} \sup_{0 \le s \le T} \sup_{0 < t - s < h} P\{\|A_t^s\| > \varepsilon / \mathfrak{F}_s^0\} = 0,$$

then the theorem remains in force if the equalities (1) and (2) hold with probability 1, and the integrals are understood as the corresponding limits in the sense of convergence in probability.

2. Let now a Markov process x_t be defined in some space X, and let $\mathfrak{F}_t^s$ denote the minimal σ-algebra with respect to which the x_u, $u \in [s, t]$, are measurable. With every multiplicative family of ope-

with the operators Π_t^s one can associate a conditionally Markov process y_t with values in Y so that the pair (x_t, y_t) is also a Markov process and

$$\mathbf{M}\{f(y(t)) / \mathfrak{F}_t^s,\ y(s)\} = \Pi_t^s f(y(s))$$

(the question of adjoining a conditionally Markov process to a Markov process was considered by A. D. Venttsel' in (^{1})).

In the case when the theorem is applicable, the description of all multiplicative families reduces to the description of additive families A_t^s. Since such a family is determined by a function $\alpha_t^s(y, A)$, $y \in Y$, $A \in \mathfrak{B}$, it is sufficient to specify this function. From the conditions imposed on A_t^s it follows that, for $y \in \bar{A}$, the quantity $\alpha_t^s(y, A)$ is a nonnegative additive functional of the process x_t; the same functional will be $-\alpha_t^s(y, \{y\}) = \alpha_t^s(y, Y - \{y\})$; $\{y\}$ is the set consisting of the point y. In the case of homogeneous processes, nonnegative additive homogeneous functionals are completely described (see (^{2}), Ch. 6). This makes it possible to describe all homogeneous M-families of operators by means of additive functionals (instead of condition M-3 one may impose condition M-4).

In the simplest case, when Y is a finite set $\{y_1, \dots, y_n\}$, the probability operator Π_t^s is determined by a stochastic matrix $\|p_t^s(i, j)\|$, $i, j = 1, \dots, n$. If the functions $p_t^s(i, j)$ satisfy the condition: for every $\varepsilon > 0$, with probability 1,

$$\lim_{h \to 0} \sup_{s \leq T} \sup_{0 < t-s < h} \mathbf{P}\{|p_t^s(i, j) - \delta_{ij}| > \varepsilon / x_s\} = 0,$$

where δ_{ij} are the elements of the identity matrix, then there exist nonnegative additive functionals $\alpha_t^s(i, j)$, $i \neq j$, $i, j = 1, \dots, n$, such that

$$d_s p_t^s(i, j) = \sum_{k \neq i} [p_t^s(k, j) - p_t^s(i, j)]\, d_s \alpha_t^s(i, k).$$

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.

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