MATHEMATICS

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Abstract

Full Text

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ON MONOTONE ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician S. N. Bernstein, 16 XI 1963)

We consider the class B'_σ of entire functions

$$\varphi_\sigma(z) = \sum_{k=0}^{\infty} c_k z^k \tag{1}$$

of degree σ, real on the real axis, monotonically increasing and bounded on this axis. Let

$$\omega_j(\varphi_\sigma) \equiv \sum_{k=1}^{\infty} c_k \alpha_{kj} \quad (j = 1, 2, \dots, p)$$

be linear functionals defined on the set B'_σ. The relations

$$\omega_j(\varphi_\sigma) = A_j \quad (j = 1, 2, \dots, p), \tag{2}$$

where A_j are given real numbers (with $\exists A_j \neq 0$), will be called admissible if they are compatible and do not contradict the monotonicity of $\varphi_\sigma(x)$.

By the norm of the function $\varphi_\sigma(x)$ we shall mean

$$\|\varphi_\sigma(x)\| = \max_{-\infty<x<\infty} |\varphi_\sigma(x)| \tag{3}$$

under the fulfillment of conditions (2). A function $\varphi^*_\sigma(x)$ is called an extremal function of the problem if it has the minimal norm (i.e., if it deviates least from zero in the Chebyshev sense on the entire real axis).

Then the following is true.

Theorem 1. *Among the extremal functions $\varphi_\sigma(z) \in B'_\sigma$ subject to the relations (2) for $p \leq 2$, there exists a function $\varphi^*_\sigma(z)$ such that*

$$\frac{d}{dx}\varphi^*_\sigma(x) = [\psi_{\sigma/2}(x)]^2, \tag{4}$$

where $\psi_{\sigma/2}(x)$ is some entire function of degree $\sigma/2$.

This theorem is analogous to the well-known theorem of S. N. Bernstein (^{1}) on nonnegative trigonometric polynomials (and also to our (^{2}) theorem for monotone algebraic polynomials). Using it, we shall show here the solution of some extremal problems in the class B'_σ.

Without loss of generality, one may assume that the functions $\varphi_\sigma(x)$ are odd (see (^{3})); consequently,

$$\varphi^*_\sigma(x) = \int_0^x [\psi_{\sigma/2}(t)]^2\,dt, \tag{5}$$

and its norm is

$$\|\varphi^*_\sigma(x)\| \equiv E_{\varphi^*_\sigma} = \int_0^\infty [\psi_{\sigma/2}(t)]^2\,dt. \tag{6}$$

By the well-known Wiener—Paley theorem, the function $\psi_{\sigma/2}(x)$ is representable in the form

$$\psi_{\sigma/2}(x) = \int_{-\sigma/2}^{\sigma/2} f(t)e^{ixt}\,dt, \tag{7}$$

where

$$f(t) \in L_2[-\sigma/2, \sigma/2], \qquad \int_0^\infty |\psi_{\sigma/2}(t)|^2\,dt = \pi\int_{-\sigma/2}^{\sigma/2} \left|f(t)^{ixt}dt\right|^2 \tag{8}$$

Consequently,

$$E_{\varphi^*_\sigma} = \pi\int_{-\sigma/2}^{\sigma/2} |f(t)|^2\,dt. \tag{9}$$

Thus, the problem of finding the extremal function $\varphi^*_\sigma(x)$ in the class B'_σ under conditions (2) (as well as its deviation $E_{\varphi^*_\sigma}$) is reduced to minimizing the integral

$$\int_{-\sigma/2}^{\sigma/2} |f(t)|^2\,dt \tag{10}$$

under the corresponding conditions imposed on the function $f(t)$, following from (2).

Since $f(t) \in L_2[-\sigma/2, \sigma/2]$, from the sequence $\{\widehat{\mathcal{P}}_n(x)\}$ of normalized Legendre polynomials on $[-\sigma/2, \sigma/2]$ one can select a subsequence $\{\widehat{\mathcal{P}}_{n_k}(x)\}$ such that

$$f(x) = \sum_{k=0}^{\infty} a_{n_k} \widehat{\mathcal{P}}_{n_k}(x) \tag{11}$$

uniformly almost everywhere on $[-\sigma/2, \sigma/2]$. Consequently,

$$E_{\varphi_\sigma^*} = \pi \sum_{k=0}^{\infty} a_{n_k}^2, \tag{9'}$$

and the problem is reduced to minimizing the sum $\sum_{k=0}^{\infty} a_{n_k}^2$ under the corresponding constraints (following from (2)) imposed on a_{n_k} (not always linear).

Let us consider several extremal problems, for the solution of which we shall use Theorem 1.

Problem A (S. N. Bernstein (³)). Find the extremal function $\varphi_\sigma^*(x) \in B'_\sigma$ and its deviation from zero $E_{\varphi_\sigma^*}$, if $p = 1$ and

$$\omega(\varphi_\sigma) = \varphi'_\sigma(0) = 1. \tag{2^A}$$

Since Theorem 1 is applicable to the problem, expanding the function $f(x)$ in the series (11) and minimizing the right-hand side of expression $(9')$ under condition (2^A), we find

$$n_0 = 0 \qquad a_0 = \frac{1}{\sqrt{\sigma}}, \qquad a_{n_k} = 0 \quad (k = 1, 2, \ldots),$$

i.e.

$$f(x) = \frac{1}{\sigma}.$$

Consequently,

$$\psi_{\sigma/2}(x) = \frac{1}{\sigma} \int_{-\sigma/2}^{\sigma/2} e^{ixt}\, dt = \frac{\sin(\sigma x/2)}{\sigma x/2},$$

$$\varphi_\sigma^*(x) = \int_0^x \left(\frac{\sin(\sigma t/2)}{\sigma t/2}\right)^2 dt, \qquad E_{\varphi_\sigma^*} = \frac{\pi}{\sigma}$$

(this result was obtained by another method in (³)).

Problem B. Let $p = 1$,

$$\omega(\varphi_\sigma) \equiv \varphi_\sigma'''(0) = A. \qquad (2\ )$$

It is required (as in Problem A) to find the extremal function $\varphi_\sigma^*(x) \in B$ and its deviation from zero $E_{\varphi_\sigma^*}$.

Using Theorem 1 and carrying out computations analogous to those applied to Problem A (expanding $f(x)$ in the series (11) and minimizing the right-hand side of expression (9) under condition (2)), we find:

$$n_0 = 0, \qquad a_0^2 = \frac{2\sqrt{5}}{\sigma^3}|A|, \qquad n_1 = 2, \qquad a_2^2 = \frac{7\sqrt{5} - 15}{\sigma^3}|A|,$$

$$a_{n_k} = 0 \quad (k = 2, 3, \dots), \qquad (12)$$

$$E_{\varphi_\sigma^*} = \frac{9\sqrt{5} - 15}{\sigma^3}\,\pi|A|$$

(the expression for $\varphi_\sigma^*(x)$ is not given here because of its unwieldiness, but it is easy to find, taking into account (7), (11), and (12)).

Hence follows

Theorem 3. If $\varphi_\sigma(x) \in B_\sigma'$ and $M = \max\limits_{-\infty<x<\infty} |\varphi_\sigma(x)|$, then

$$\max_{-\infty<x<\infty} |\varphi_\sigma'''(x)| \leq \frac{3\sqrt{5} + 5}{60}\,\frac{\sigma^3}{\pi}\,M,$$

and this inequality is sharp (equality is attained on the function $\varphi_\sigma^*(x)$ of Problem B).

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