MATHEMATICS

M. G. KREIN, G. K. LANGER

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. G. KREIN, G. K. LANGER

ON THE THEORY OF QUADRATIC PENCILS OF SELF-ADJOINT OPERATORS

(Presented by Academician L. S. Pontryagin, 21 X 1963)

Various problems of mathematical physics lead to the study of the spectral properties of the quadratic pencil $L(\lambda) = \lambda^2 I + \lambda B + C$, where B, C are certain closed operators (I is the identity operator) acting in a Hilbert space $\mathfrak{H}$.

The methods set forth below for investigating the pencil $L(\lambda)$, as well as a number of propositions obtained with their aid, apparently are new even for a quadratic matrix pencil.

1. Denote by $\mathfrak{R}$ the ring of all bounded operators in $\mathfrak{H}$, and by $\mathfrak{C}_\infty$ the ideal of all completely continuous operators from $\mathfrak{R}$.

In what follows we assume that $B = B^*$ (i.e. B is any self-adjoint operator) and that $\mathfrak{D}(B) \subset \mathfrak{D}(C)$.

1°. For $\lambda \neq 0$ the operator $L(\lambda)$ is defined on $\mathfrak{D}(B)$, and the set $\rho(L)$ of all regular points λ ($\neq 0$) of the pencil (i.e. points λ ($\neq 0$) for which there exists a continuous inverse operator $L^{-1}(\lambda) \in \mathfrak{R}$) is an open set.

The complement $\sigma(L)$ in the complex plane to the set $\rho(L)$ is called the spectrum of the pencil $L(\lambda)$.

2°. If $C = C^*$, then $\sigma(L) = \overline{\sigma(L)}$ (i.e. the spectrum $\sigma(L)$ is symmetric with respect to the real axis). For a nonreal isolated eigenvalue λ of the pencil $L(\lambda)$, the root subspaces* of the pencil $\mathfrak{L}_\lambda(L)$ and $\mathfrak{L}_{\bar\lambda}(L)$ have the same dimension ($\leqslant \infty$) (and, moreover, in a certain sense have the same structure).

3°. If B and C are nonnegative self-adjoint operators ($B, C \geqslant 0$), then the spectrum $\sigma(L)$ lies in the left half-plane.

4°. If the operator C is B-completely continuous (^{2}), then every point $\lambda_0 \in \sigma(L)$, $\lambda_0 \neq 0$, not belonging to the condensation spectrum** of the operator B, is an isolated point of the spectrum $\sigma(L)$, and moreover an eigenvalue of finite algebraic multiplicity (i.e. the root subspace $\mathfrak{L}_{\lambda_0}(L)$ has finite dimension and $L(\lambda_0)$ is a Φ-operator (^{2})).

At the basis of our further investigation lies the study of the quadratic operator equation:

$$Z^2 + BZ + C = 0. \tag{1}$$

If $C \in \mathfrak{R}$ and $Z_0(Z_0 \in \mathfrak{R},\ \mathfrak{R}(Z_0) \subset \mathfrak{D}(B))$ is some root of equation (1), then

$$\lambda^2 I + \lambda B + C = (\lambda I - \hat{Z}_0)(\lambda I - Z_0),$$

where $\hat{Z}_0 = -B - Z_0$. If $B = B^*$ and $C = C^* \in \mathfrak{R}$, then together with Z_0 the operator $\hat{Z}_0^*$ will also be a root of equation (1) (in the case of unbounded B, in the sense that $Z^2x + BZx + Cx = 0$ for every $x \in \mathfrak{D}(B^2)$). It may happen that $\hat{Z}_0^* = Z_0$ (see Theorem 2).

---

* If λ_0 is an eigenvalue of the pencil $L(\lambda)$, then the root subspace $\mathfrak{L}_{\lambda_0}(L)$ of the pencil is the linear span of all eigenvectors and associated vectors of the pencil $L(\lambda)$ corresponding to the number λ_0 (see ([1])).

** The condensation spectrum of the operator B comprises all limit points of its spectrum $\sigma(B)$ and its eigenvalues of infinite multiplicity.

Theorem 1. Let $B = B^*$ ($\|B\| \leqslant \infty$), and let $C \in \mathfrak{R}$ be a nonnegative operator having a B-completely continuous nonnegative root $C^{1/2}$. Then, for any partition of the nonreal spectrum $\sigma_0(L)$ of the pencil $L(\lambda)$ into two disjoint parts Λ and $\overline{\Lambda} = \sigma_0(L) \setminus \Lambda$, symmetrically situated with respect to the real axis, equation (1) has a root Z_Λ ($Z_\Lambda \in \mathfrak{R},\ \mathfrak{R}(Z_\Lambda) \subset \mathfrak{D}(B)$) with the properties: 1) $Z_\Lambda^* Z_\Lambda \leqslant C$, 2) the nonreal part of the spectrum $\sigma(Z_\Lambda)$ coincides with Λ.

Let us note that, under the conditions of the theorem, properties 1) and 2) of the root Z_Λ imply the property: 3) $\sigma(Z_\Lambda) \subset \sigma(L) \cup \{0\}$, and for every $\lambda_0 \in \Lambda$ the root subspace $\mathfrak{L}_{\lambda_0}(Z_\Lambda)$ coincides with the root subspace $\mathfrak{L}_{\lambda_0}(L)$ of the pencil $L(\lambda)$.

We give some explanations for the proof of the theorem. Take two copies $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of the space $\mathfrak{H}$ and form their direct orthogonal sum $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Put $J = P_1 - P_2$, where P_j ($j = 1, 2$) is the projector orthogonally projecting $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}_j$. A subspace $M \subset \widetilde{\mathfrak{H}}$ is called **maximal J-nonnegative** if it is a maximal subspace with the property $(J\tilde{x}, \tilde{x}) \geqslant 0$ for $\tilde{x} \in M$. Every such subspace M is given by its angular operator K according to the rule: $M = \{x_1 + Kx_1 :\ x_1 \in \mathfrak{H}_1\}$, where K is a certain nonexpanding operator ($\|K\| \leqslant 1$) mapping $\mathfrak{H}_1$ into $\mathfrak{H}_2$ ([3,4]). Consider the operator $H = \|H_{jk}\|_1^2$ ($H_{jk} = P_{jHP}k;\ j, k = 1, 2$), for which

$$H_{11} = 0, \qquad H_{12} = C^{1/2}, \qquad H_{21} = -C^{1/2}, \qquad H_{22} = -B.$$

This operator is J-self-adjoint (i.e. $(JH)^* = JH$). The conditions of the theorem mean that Theorem 4 of ([5]) is applicable to H, by virtue of which H has at least one maximal J-nonnegative subspace M ($\subset \mathfrak{D}(H)$), invariant with respect to H. It is easy to verify that the angular operator K of every such subspace M satisfies the relation

$$KC^{1/2}K + BK + C^{1/2} = 0,$$

and therefore $Z_0 = KC^{1/2}$ satisfies equation (1) and has property 1).

A more complete use of Theorem 4 of ([5]) makes it possible to establish the existence of a root Z_Λ which also has property 2).

Theorem 2. If, under the conditions of Theorem 1, the linear manifolds $\mathfrak{L}_\lambda(L)$ $(\lambda \in \Lambda)$ form a complete system in $\mathfrak{H}$, then the root Z_Λ is uniquely determined by the given Λ. In this case $Z_\Lambda \in \mathfrak{S}_\infty$, and the factorization

$$\lambda^2 I + \lambda B + C = (\lambda I - Z_\Lambda^*)(\lambda I - Z_\Lambda),$$

holds, and consequently

$$B = -(Z_\Lambda + Z_\Lambda^*) \in \mathfrak{S}_\infty \quad \text{and} \quad C = Z_\Lambda^* Z_\Lambda \in \mathfrak{S}_\infty.$$

2. If $B \in \mathfrak{R}$, then the condition that $C^{1/2}$ be B-completely continuous is equivalent to the condition $C \in \mathfrak{S}_\infty$. In what follows, we consider mainly the case when $C \in \mathfrak{S}_\infty,\ C \geqslant 0$. In this case, for any $B = B^*$, all the conditions of Theorem 1 are satisfied,* and the root Z_Λ given by this theorem, by virtue of its property 1), will be completely continuous. Let

$$\lambda_1(C) \geqslant \lambda_2(C) \geqslant \cdots$$

be the successive eigenvalues of the operator C, and let $\{\lambda_j(Z_\Lambda)\}_1^N$ $(N \leqslant \infty)$ be the complete sequence of all nonzero eigenvalues (counting their algebraic multiplicities) of the operator Z_Λ, arranged in some order of nondecreasing moduli. By a result of T. Weyl ([8]), from property 1) of the root Z_Λ it follows that for any function $f(r)$ $(0 \le r < \infty;\ f(+0) = 0)$, to which there corresponds a function $f(e^t)$ convex downward $(-\infty < t < \infty)$, the inequalities

$$\sum_{j=1}^{n} f(|\lambda_j(Z_\Lambda)|) \leqslant \sum_{j=1}^{n} f(\sqrt{\lambda_j(C)}) \qquad (n = 1, 2, \dots, N). \tag{2}$$

* Let us note that in this important case Theorem 1 is proved in the same way on the basis of the results of ([6], [7]), earlier than ([5]).

It turns out that if C $(\in \mathfrak{S}_\infty)$ is a positive operator ($(Cx, x) > 0$ for $x \neq 0$), $N = \infty$, and for some strictly convex function $f(e^t)$ the right-hand side in (2) is finite for $n = \infty$, then for this function the sign $=$ occurs in (2) for $n = \infty$ if and only if the operator B is permutable with the operator C and the condition $(Bx, x)^2 \le 4(x, x)(Cx, x)$ is satisfied.

3. We shall be interested in the condition of weak damping of the pencil $L(\lambda)$:

$$(Bx, x)^2 < 4(x, x)(Cx, x) \qquad (x \neq 0). \tag{3}$$

It is easy to see that if this condition is satisfied (for $x \in \mathfrak{D}(B)$, $x \neq 0$), then, together with the operator $C \in \mathfrak{S}_\infty$, also $B \in \mathfrak{S}_\infty$ ($\mathfrak{D}(B) = \mathfrak{H}$). Condition (3) is equivalent to the condition of positivity of the operator $L(\lambda)$ for every real λ. If it is known in advance that $B = B^* \in \mathfrak{S}_\infty$ and $C = C^* \in \mathfrak{S}_\infty$, then the fulfillment of (3) is a necessary and sufficient condition that the spectrum $\sigma(L)$ contain no real points $\lambda \neq 0$. In this case $\sigma(L)$ consists of $\Lambda \cup \overline{\Lambda}$ and the point $\lambda = 0$, and inequalities (2) give a certain characterization of the distribution of the moduli of the eigenvalues of the whole spectrum $\sigma(L)$.

Theorem 3. Let B be a nonnegative nuclear operator ($\operatorname{Sp} B < \infty$), and let condition (3) be satisfied. Then the relation

$$-\sum_j \operatorname{Re} \lambda_j \leq \operatorname{Sp} B, \tag{4}$$

holds, where the summation extends over all eigenvalues (counting their multiplicities) of the pencil $L(\lambda)$. Equality in relation (4) holds if and only if the system of root vectors of the root λ is complete in $\mathfrak{H}$. This case occurs whenever $\liminf n^2 \lambda_n(C) = 0$, and, in particular, when $\operatorname{Sp}(C^{1/2}) < \infty$.

4. Let us introduce some general definitions for the pencil $L(\lambda)$ with $C \gg 0$. Let x_0 be an eigenvector of the pencil $L(\lambda)$ corresponding to the eigenvalue λ_0. Then three cases are possible: the quantity $|\lambda_0|^2$ may be equal to, less than, or greater than the ratio $(Cx_0, x_0)/(x_0, x_0)$. Corresponding to these cases, the eigenvector x_0 is called neutral, of the first kind, or of the second kind.

If λ_0 is nonreal, then the eigenvector x_0 will necessarily be neutral.

If the eigenvalue λ_0 has corresponding eigenvectors of one and the same kind (first or second), then their totality (the eigensubspace $\mathfrak{E}_{\lambda_0}(L)$) will coincide with the root subspace $\mathfrak{L}_{\lambda_0}(L)$.

We now consider a pencil $L(\lambda)$ for which the condition of strong damping is satisfied:

$$(Bx, x) > 2\sqrt{(Cx, x)(x, x)} \qquad (x \in \mathfrak{D}(B),\ x \neq 0). \tag{5}$$

This condition entails the uniform positivity of the operator B, i.e. $\inf[(Bx, x)/(x, x)] > 0$. When it is fulfilled, the spectrum of the pencil $L(\lambda)$ is situated on the negative semiaxis and the pencil has no neutral eigenvectors.

Theorem 4. Let the positive operator $C \in \mathfrak{S}_\infty$, $B \in \mathfrak{R}$, and let condition (5) be satisfied.

Then

1°. The quadratic equation (1) has one and only one root Z_1 (Z_2) with the property $Z_1^* Z_1 \leq C$ ($Z_2^* Z_2 \geq C$), and in addition

$$Z_2 = -B - Z_1^*, \qquad Z_2^* Z_1 = C.$$

2°. The operator

$$S = B + Z_1 + Z_1^* = -(B + Z_2 + Z_2^*) = Z_1 - Z_2$$

is a uniformly positive operator.

3°. The roots Z_k ($k = 1, 2$) are symmetrized by the operator S ($SZ_k = Z_k^* S,\ k = 1, 2$), and, moreover, they are similar to negative operators; furthermore,

$$(SZ_2 y, y) < (SZ_1 x, x) < 0$$

for any $x, y \in \mathfrak{H}$ with $\|x\| = \|y\| = 1$.

4°. The spectrum

$$\sigma(L) = \sigma(Z_1) \cup \sigma(Z_2),$$

and

$$\max\{\lambda : \lambda \in \sigma(Z_2)\} \leq \min\{\lambda : \lambda \in \sigma(Z_1)\}.$$

5°. The eigenvectors of the root Z_1 (Z_2) exhaust all eigenvectors of the first (second) kind of the pencil $L(\lambda)$.

Theorem 4 can easily be reformulated for the case of nonnegative $C \in \mathfrak{S}_\infty$, and also for the case of an unbounded B ($= B^*$).

We also omit possible generalizations of Theorem 4 to the case of bounded operators C (≥ 0) with a B-completely continuous square root $C^{1/2}$.

Let us note that the task of establishing Theorem 4 was somewhat facilitated for the authors by the original work ([9]), in which, by other methods, oscillations of strongly damped systems with a finite number of degrees of freedom were investigated.

In conclusion we point out that our methods make it possible to establish a number of propositions for the pencil $L(\lambda)$ also in the case of an indefinite operator $C = C^*$. In this connection we note that the case of a nonpositive C admits treatment within the framework of the usual spectral theory of self-adjoint operators ([10,11]); however, even in this case the study of the quadratic equation (1) leads to more general and precise results.

A more detailed exposition of the present results will be given in the "Proceedings of the International Symposium on Applications of the Theory of Functions of a

Complex Variable in Continuum Mechanics," where the authors will also dwell on questions of the twofold completeness of eigenvectors and associated vectors of the pencil

$$\lambda^2 L(\lambda^{-1}) = I + \lambda B + \lambda^2 C,$$

studied under other assumptions by M. V. Keldysh ([1]).

Odessa Civil Engineering Institute

Dresden Technical University
Dresden, GDR

Received
17 X 1963

REFERENCES

1. M. V. Keldysh, DAN, **77**, No. 1 (1951).
2. I. Ts. Gokhberg, M. G. Krein, UMN, **12**, 2 (74), 43 (1957).
3. R. S. Phillips, Trans. Am. Math. Soc., **90**, 193 (1959).
4. Yu. P. Ginzburg, DAN, **139**, No. 4, 775 (1961).
5. M. G. Krein, DAN, **154**, No. 5 (1964).
6. H. Langer, Math. Ann., **146**, 60 (1962).
7. H. Langer, Math. Ann., **152**, 434 (1963).
8. H. Weyl, Proc. Nat. Acad. Sci., **35**, 408 (1949).
9. R. J. Duffin, J. Rat. Mech. and Analysis, **4**, No. 2, 221 (1955).
10. D. F. Kharazov, Matem. sborn., **42** (84), 2, 129 (1957).
11. R. H. Müller, Arch. Math., **12**, 4, 308 (1961); Math. Zs., **70**, 381 (1959).

Note: Figure translations are in progress. See original paper for figures.