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Abstract

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ON THE CONVERGENCE OF ORTHOGONAL SERIES

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I. In this note we obtain some new sufficient conditions for the convergence of orthogonal series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x), \tag{1}$$

expressed in terms of the coefficients of the series. As is known, the fundamental theorem of Menshov–Rademacher ([1]) asserts that the condition

$$\sum_{n=1}^{\infty} a_n^2 \log^2 n < \infty \tag{2}$$

is sufficient for the convergence almost everywhere on $[c, d]$ of the series (1) (if $\{\varphi_n(x)\}$ is an orthonormal system (o.n.s.) on $[c, d]$ with respect to some measure $\mu(x)$, $-\infty \le c < d \le \infty$). D. E. Menshov proved that even for the class of o.n.s. uniformly bounded on $[c, d]$, $\log^2 n$ is the exact Weyl multiplier. K. Tandori ([1]) in 1958 established, developing the method of D. E. Menshov, a stronger proposition: if $|a_n| \downarrow 0$, then condition (2) is also necessary in the class of uniformly bounded o.n.s.

In a recent work K. Tandori ([2]) proved the following theorem: in order that all series (1) with respect to o.n.s. on $[0, 1]$ converge almost everywhere, it is necessary and sufficient that for all sequences $n_1 < n_2 < n_3 < \cdots$ the conditions

$$\sum_{s=0}^{\infty} M(a_{n_s+1}, \dots, a_{n_{s+1}}) < \infty,$$

be satisfied, where

$$M(b_1, \dots, b_N) = \sup \int_0^1 \left(\max_{1 \le i \le j \le N} |b_i \varphi_i(x) + \cdots + b_j \varphi_j(x)| \right)^2 dx$$

and the supremum is taken over all o.n.s. $\{\varphi_k(x)\}$ on $[0,1]$.

However, the problem of necessary and sufficient convergence conditions effectively expressed in terms of the coefficients $\{a_n\}$ remains open. By virtue of the theorem just cited, this problem is equivalent to an effective estimate of the quantities $M(b_1, \ldots, b_N)$ from above and below for arbitrary coefficients $\{b_k\}$.

The papers ([3,4]) contain some results concerning estimates of M from below; at the same time the Menshov–Rademacher estimate

$$M(b_1, \ldots, b_N) \le B_1 \left(\sum_{k=1}^{N} b_k^2\right) \log^2(N+1) \tag{3}$$

has not yet been improved. (Here and below, B_i denote absolute constants, $B_i > 0$.)

In the present note estimate (3) is strengthened; this makes it possible to generalize many results of the theory of orthogonal series, especially for systems uniformly bounded in the aggregate. The main idea of the proposed method consists in a dyadic decomposition of the interval $[1,n]$ into parts, generally speaking, of unequal length (in contrast to the classical method). The points of division are determined by a certain function $\Phi(a)$, depending on the coefficients under consideration.

II. Let the function $\Phi(a) = \Phi(a_1, \ldots, a_n)$ be defined for all $a = \{a_i\}_{i=1}^n$ and satisfy the conditions:

1) $\Phi(a) + \Phi(b) = \Phi(a+b)$, if $\sum_{i=1}^{n} |a_i b_i| = 0$;
2) $\Phi(a) \ge 0, \qquad \Phi(0) = 0$;
3) $\Phi(\lambda a) = |\lambda|^\alpha \Phi(a)$ for all $\lambda \ne 0$ and for some α, $-\infty < \alpha < \infty$.

In what follows it is assumed that $\{\varphi_n(x)\}$ is an orthonormal system on $[c,d]$ with respect to some measure $\mu(x)$; convergence almost everywhere is considered with respect to this measure. If $b = \{b_k\}_{k=1}^n$, then

$$\|b\| = \left(\sum_{k=1}^{n} b_k^2\right)^{1/2};$$

for $f \in L^2_\mu[c,d]$

$$\|f\| = \left(\int_c^d |f(x)|^2 \, d\mu(x)\right)^{1/2}.$$

Lemma. Let $\Phi(a)$ satisfy conditions 1)–3), and let $b = \{b_k\}_{k=1}^n$ be some numbers, $b \ne 0$.

There exist a representation

$$b = b' + b'' \qquad \left(b' = \{b'_k\}_{k=1}^n, \quad b'' = \{b''_k\}_{k=1}^n, \quad \sum_{k=1}^n |b'_k b''_k| = 0\right),$$

and a sequence $0 = n_1 < n_2 < \dots < n_r \le n$ such that:

$$\|\delta_1(x)\| + \|\delta_2(x)\| \le B_1 \|b\| \left|\log\left[\frac{\Phi(b)}{\|b\|^\alpha} + B_2\right]\right|, \tag{4}$$

$$\Phi(0, \dots, 0, b'_{n_i+1}, \dots, b'_{n_{i+1}}, 0, \dots, 0) \le \|b\|^\alpha \qquad (i = 1, 2, \dots, r-1). \tag{5}$$

Here

$$\delta_1(x) = \sup_{1 \le j < r} \left|\sum_{k=1}^{n_j} b'_k \varphi_k(x)\right|, \qquad \delta_2(x) = \sup_{1 \le m < n} \left|\sum_{k=1}^{m} b''_k \varphi_k(x)\right|.$$

Corollary 1. The assertion of the lemma is valid, in particular, if

$$\Phi(b) = \sum_{i=1}^n |b_i|^\alpha M_i, \qquad M_i \ge 0, \quad -\infty < \alpha < \infty \quad (0^0 = 0).$$

For $M_i = 1$ and $\alpha = 0$ we obtain the Menshov-Rademacher lemma ($b' = b,\ b'' = 0$).

Remark. The conditions of the lemma can be weakened by assuming $\Phi(a+b) \ge \Phi(a) + \Phi(b)$, $(|a|, |b|) = 0$, $\Phi(\lambda a) \le |\lambda|^\alpha \Phi(a)$, and adding some inessential conditions.

We shall now indicate some applications of the main lemma. Their number could be increased.

Theorem 1. Let $\{\varphi_n(x)\}$ be an orthonormal system on $[c, d]$, and let $M_k > 0$ be constants such that $|\varphi_n(x)| \le M_n$ outside $E_n \subset [c, d]$, and for every set E of finite μ-measure

$$\sum_{n=1}^\infty \mu(E_n \cap E) < \infty.$$

Then, for any $\{a_k\}$, from the condition

$$\sum_{n=1}^\infty a_n^2 \log^2 \left(\sum_{k=1}^n |a_k| M_k + 1\right) < \infty \tag{6}$$

there follows the convergence of the series $\sum_{n=1}^\infty a_n \varphi_n(x)$ almost everywhere.

Corollary 2. If $M_n > 0$ is a sequence such that

$$\sum_{k=1}^\infty M_k^{-2} < \infty,$$

then from condition (6) there follows almost everywhere convergence of the series (1) for all orthonormal systems $\{\varphi_n(x)\}$. In particular, one may take $M_k = \sqrt{k}\ln k$ or $M_k = k^{1/2+\varepsilon}$, $\varepsilon > 0$.

Corollary 3. Since from (6), for $M_k = k^{1/2+\varepsilon}$, it follows that $|a_k| = O(1)$, and

$$\log\left(1+\sum_{k=1}^{n}|a_k|M_k\right)\le C\log n,$$

Theorem 1 contains the Men'shov-Rademacher theorem for arbitrary o.n.s. and, as is easy to see, is not equivalent to it.

The most convex results are obtained by means of the proposed method for o.n.s. that are uniformly bounded. In what follows we shall concentrate our attention on these systems.

III. We shall further assume that the o.n.s. $\{\varphi_n(x)\}$ is such that

$$|\varphi_n(x)|\le M \quad (n=1,2,...), \qquad M>0, \tag{7}$$

almost everywhere on $[c,d]$ (with respect to μ). Let us reformulate Theorem 1 for this case.

Theorem 2. If

$$\sum_{k=1}^{\infty} a_k^2 \log^2\left(\sum_{s=1}^{k}|a_s|+1\right)<\infty, \tag{8}$$

then all series $\sum_{n=1}^{\infty} a_n\varphi_n(x)$ over systems (7) converge almost everywhere.

Theorem 3. Let

$$b_k^2=\sum_{s=n_k+1}^{n_{k+1}} a_s^2, \qquad 0=n_0<n_1<n_2<\cdots.$$

If the series

$$\sum_{m=1}^{\infty} b_m^2 \log^2\left(\sum_{k=1}^{m}|b_k|+1\right)$$

converges, then for all systems (7) the partial sums $S_{n_k}(x)$ converge almost everywhere.

For $n_k = 2^k$ one obtains a criterion for convergence of the sequence $S_{2^k}(x)$ and (C,α)-summability, generalizing the well-known theorem of Men' shov ([1]).

From the lemma one also automatically obtains the following strengthenings of known estimates (see ([1])) of partial sums for orthogonal systems (7):

1°. If

$$\sum_{n=1}^{\infty} a_n^2 < \infty,$$

then almost everywhere

$$S_n(x) = o\left(\log\left(\sum_{k=1}^{n} |a_k| + 1\right)\right)$$

(instead of $S_n(x) = o(\log n)$).

2°. If

$$\sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n \sum_{k=1}^{n} a_k^2} < \infty,$$

then

$$S_n(x) = o\left(\log\left(\sum_{k=1}^{n} |a_k| + 1\right)\right)\left(\lambda_n \sum_{k=1}^{n} a_k^2\right)^{1/2}.$$

3°. If

$$\sum_{n=1}^{\infty} a_n^2 \log^2\left(\sum_{k=1}^{n} |a_k| + 1\right)\lambda^2(n) < \infty, \qquad \lambda(n) \uparrow \infty, \qquad f = \sum_{k=1}^{\infty} a_k \varphi_k,$$

then

$$|S_n(x) - f(x)| = o\left(\frac{1}{\lambda(n)}\right)$$

almost everywhere.

Let $\lambda_n \uparrow \infty$, $\lambda_n \geq 1$. We shall call a series **generalized** (n_k, λ_{n_k})**-lacunary** if

$$N_k(a) = \sum_{s=n_k+1}^{n_{k+1}} |a_s| \Big/ \left(\sum_{s=n_k+1}^{n_{k+1}} |a_s|^2\right)^{1/2} = O(\lambda_{n_k}) \qquad \left(\frac{0}{0} = 0\right). \tag{9}$$

This definition includes the usual definitions of lacunarity (see ([1])), since if the number of nonzero terms between n_k and n_{k+1} is $O(\mu_{n_k})$, then $N_k(a) = O(\lambda_{n_k})$, $\lambda_{n_k} = \sqrt{\mu_{n_k}}$. On the other hand, many series whose coefficients are not zero, but "almost all" coefficients are small, are (n_k, λ_{n_k})-lacunary.

Theorem 4. If $S_{n_k}(x)$ for the system (7) converges almost everywhere, the series (1) is generalized (n_k, λ_{n_k})-lacunary, and

$$\sum_{k=1}^{\infty} a_k^2 \log^2 \lambda_k < \infty,$$

then the series (1) converges almost everywhere.

In conclusion we make several remarks.

1. The results of the paper extend to orthonormalized kernels (see (5)). In particular, for Fourier integrals we obtain the following theorem:

If $b(t)$ is a measurable function on $(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} |b(t)|^2 \log^2 \left(\int_{-t}^{t} |b(u)|\, du + 1 \right) dt < \infty,$$

then for almost all $x \in (-\infty, \infty)$ there exists

$$\lim_{a \to \infty} \int_{-a}^{a} e^{itx} b(t)\, dt.$$

We note that definition (9) suggests natural ways of considering "lacunary" integrals with respect to orthonormalized kernels.

2. Both for trigonometric series and for Fourier integrals, the results given yield new sufficient conditions for convergence almost everywhere, distinct from the Kolmogorov–Seliverstov–Plessner conditions. It would be interesting to determine whether, in this case, one cannot replace

$$\log^2 \left(\sum_{k=1}^{n} |a_k| + 1 \right) \quad \text{by} \quad \log \left(\sum_{k=1}^{n} |a_k| + 1 \right).$$

3. Theorems 1-4 and their corollaries are automatically carried over to Hilbert (in the sense of N. K. Bari) systems of functions. In particular, they are valid for the systems $\{\varphi(nx)\}$, where

$$\varphi(x) = \sum_{-\infty}^{\infty} A_k e^{ikx}, \qquad \sum |A_k| < \infty, \qquad A_0 = 0$$

(for example, $\varphi(x) \in \operatorname{Lip} \alpha,\ \alpha > 1/2$).

4. From K. Tandori' s theorem cited at the beginning of the paper it follows that, for monotone $\{|a_k|\}$, conditions (2) and (8) are equivalent; therefore, when $|a_k| \downarrow 0$, condition (8) is necessary and sufficient for the convergence almost everywhere of all series (1) with respect to an o.n.s. $\{\varphi_n(x)\}$ that is bounded in the aggregate. However, we have not yet succeeded in establishing whether condition (8) is necessary for the convergence of series (1) for arbitrary $\{a_k\}$ in the class of all bounded o.n.s.

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Note: Figure translations are in progress. See original paper for figures.