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1. We consider the equation

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Abstract

Full Text

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ON THE DISTRIBUTION OF ZEROS OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION

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1. We consider the equation

$$Lx \equiv x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_{n-1}(t)x' + p_n(t)x = 0 \quad (a \leqslant t \leqslant b). \qquad (1)$$

The coefficients $p_1, \dots, p_n$ are real and, for simplicity, are assumed continuous. As usual, with the appropriate reservations (the equation is satisfied almost everywhere, $x^{(n-1)}$ is absolutely continuous), all results remain valid for summable coefficients. Comparisons with terms of the form $(px^{(i)})^k$ will also often be considered; in this case we shall assume that the corresponding derivatives exist, although here too the required order of smoothness of the coefficients can be lowered by means of absolute continuity.

Let us introduce some notation. We shall say that $y(t)$ has $(t_1, i; t_2, k)$-zeros if $a \leqslant t_1 < t_2 \leqslant b$ and

$$y(t_1) = y'(t_1) = \dots = y^{(i-1)}(t_1) = y(t_2) = y'(t_2) = \dots = y^{(k-1)}(t_2) = 0.$$

In those cases where the values t_1, t_2 are immaterial, we shall speak of (i, k)-zeros (the requirement $a \leqslant t_1 < t_2 \leqslant b$ being retained). The zeros t_1, t_2 are called adjacent if $y(t)$ has no zeros in the interval (t_1, t_2). Everywhere below, when solutions of (1) are discussed, nontrivial solutions are meant. By L^* we denote the operator adjoint to L:

$$L^*x \equiv (-1)^n x^{(n)} + \dots - (p_{n-1}x)' + p_{nx}.$$

By $\rho_k(s)$ $(\mu_k(u))$, $1 \leqslant k \leqslant n-1$, we denote the least of those u (the greatest of those s) for which (1) has a solution with $(s, n-k; u, k)$-zeros; if for some s (u) such u (s) do not exist, then for definiteness we put $\rho_k(s) = \infty$ $(\mu_k(u) = -\infty)$. By $\rho(s)$ we denote the threshold function, i.e. the leftmost change of sign, in the interval $s < t < b$, of the Cauchy function $X(t, s)$; if $X(t, s) \geqslant 0$, then we put $\rho(s) = \infty$. By $T_{i,k}$ $(T^+_{i,k})$, where $i + k \geqslant n$, we denote the class of operators L for which (1) has no solutions with (i, k)-zeros (with adjacent (i, k)-zeros). We also set

$$T_1' = \bigcup T_{n-1,1}, \quad n = 1, 2, \dots$$

($T_{0,1}$ is the class of first-order operators). The following relations are obvious:

$$\rho(s) \geqslant \rho_1(s); \quad T_{i,1} = T^+_{i,1}, \quad T_{1,i} = T^+_{1,i};$$

$$L \in T_{n-k,k} \Longleftrightarrow (-1)^n L^* \in T_{k,n-1} \Longleftrightarrow \rho_k(s) \equiv \infty \Longleftrightarrow \mu_k(s) \equiv -\infty.$$

The operator L is called nonoscillatory if every solution of (1) has on $[a, b]$ no more than $n-1$ zeros (zeros being counted according to multiplicity); the class of nonoscillatory operators of order n (of all orders) is denoted by T_0 (T_0'). As is known,

$$T_0 = \bigcap T_{k,n-k}, \quad k = 1, 2, \dots, n-1.$$

If $\rho(s) \equiv \infty$, then L is called a Chaplygin operator; the class of Chaplygin operators of order n (of all orders) is denoted by T (T'). For $L \in T$, as is known, the comparison theorem is valid:

$$y^{(i)}(t_0) = z^{(i)}(t_0), \quad i = 0, 1, \dots, n-2, \quad y^{(n-1)}(t_0) \geqslant z^{(n-1)}(t_0), \quad Ly \geqslant Lz$$

$$\Longrightarrow \quad y(t) \geqslant z(t) \quad (a \leqslant t_0 \leqslant t \leqslant b).$$

Obviously,

$$T_0 \subset T_{n-1,1} \subset T$$

(for $n \leqslant 2$ these classes coincide),

$$T_0' \subset T_1' \subset T'.$$

We recall that in the classes introduced above, for definiteness, only operators with unit coefficient at the highest derivative are included (although, in fact, only its positivity is essential).

2. Let $q_1(t), \dots, q_m(t)$ be nonnegative; put

$$Nx = (q_1 x^{(m-1)})^{(m-1)} - \dots + (-1)^n (q_{m-1} x')' - (-1)^m q_m x.$$

Theorem 1. *Let L be a self-adjoint operator of order $n = 2m$. If $L \in T_{m,m}$, then $L - N \in T_{m,m}$. In particular, $x^{(n)} - Nx \in T_{m,m}$.*

An analogous assertion is also true for operators L, N with leading terms $(px^{(m)})^{(m)}$, $(qx^{(m)})^{(m)}$, respectively, where $p(t) > 0$, $q(t) \geq 0$. Theorem 1 follows from the following proposition.

If L is a self-adjoint operator of order $n = 2m$, then $L \in T_{m,m}$ if and only if the operator $(-1)^{mL}$ is positive definite on the subspace of functions with $(a, m; b, m)$-zeros (see (2)).

Comparison with Lemma 3 from (8) shows that for

$$x^{\mathrm{IV}} + (px')' + qx$$

the nonoscillation criterion can be formulated more effectively than in (4).

Theorem 2. *For the nonoscillation of* $x^{\mathrm{IV}} + (px')' + qx$ *it is necessary and sufficient that there exist* $p_1(t) \geq p(t)$, $q_1(t) \leq q(t)$, *and* $v(t) \not\equiv 0$ *such that: 1)* $x^{\mathrm{IV}} + (p_1 x')' + q_1 x \in T_{2,2}$*; 2)* $v(a) = v'(a) = 0$, $v(t) \geq 0$, $Lv \leq 0$ $(a \leq t \leq b)$.

Theorem 3. *Let* $n = 2m - 1$. *Then*

$$x^{(n)} + Nx \in T_{m,m-1}, \qquad x^{(n)} - Nx \in T_{m-1,m}.$$

3. Put $q_+(t) = \max\{0, q(t)\}$, $q_-(t) = \max\{0, -q(t)\}$.

Theorem 4. *If the inequality*

$$\int_a^b q_+(t)\,dt \leq 4^{2m-1}(2m-1)[(m-1)!]^2(b-a)^{1-2m},$$

is satisfied, then

$$x^{(2m)} - (-1)^m qx \in T_{m,m}.$$

Comparison of Theorem 4 with Theorem 3 from (^{8}) leads to the following nonoscillation criterion.

Theorem 5. *If the inequalities*

$$\int_a^b q_+(t)\,dt \leq \frac{384}{(b-a)^3}, \qquad \int_a^b q_-(t)\,dt \leq \frac{192}{(b-a)^3}, \tag{2}$$

are satisfied, then the operator $x^{\mathrm{IV}} + qx$ *is nonoscillatory.*

Here the first of conditions (2) ensures the inclusion

$$x^{\mathrm{IV}} + qx \in T_{3,1} \cap T_{1,3},$$

and the second, the inclusion

$$x^{\mathrm{IV}} + qx \in T_{2,2}.$$

The values of the constants in Theorems 4 and 5 cannot be improved.

4. As was shown by Mammana (^{1}), the condition $L \in T_0$ is equivalent to the representability of L on $[a,b]$ in the form

   $$L \equiv (d/dt + q_1) \cdots (d/dt + q_n),$$

   where $q_i(t)$ are continuous. From this classical theorem it follows, in particular, that T_0' has the semigroup property:

   $$L_1, L_2 \in T_0' \to L_1 L_2 \in T_0'$$

   (it is assumed that L_2 has sufficiently smooth coefficients, so that the operator $L_1 L_2$ exists). The classes T', T_1' have an analogous semigroup property; moreover,

   $$L_1, L_2 \in T' \to L_1 L_2 \in T_1'.$$

The following observation is also useful:

$$q(t) \geq 0,\ L \in T\ (T_{n-1,1}) \rightarrow Lx - qx \in T\ (T_{n-1,1});$$

if, in addition, $q(t) \not\equiv 0$ on every interval, then

$$L \in T \rightarrow Lx - qx \in T_{n-1,1}.$$

These considerations lead, for example, to the following assertion.

Theorem 6. *Let each of the numbers* $i_1,\ i_2 - i_1, \dots,\ i_{n-1} - i_{n-2}$ *be equal either to zero or to one. If* $q_0(t), q_1(t), \dots, q_{n-2}(t) \geq 0$, *then*

$$x^{(n)} - \sum_{k=1}^{n-1} \left(q_k x^{(i_k)}\right)^{(k-i_k)} - qx_0 \in T_{n-1,1}.$$

Examples.

$$x^{(n)} + p_1 x^{(n-1)} - p_2 x^{(n-2)} - \dots - p_{nx} \quad (p_2, \dots, p_n \geq 0),$$

$$x''' - (px')' - qx \quad (q \geq 0), \qquad x''' - (px)' - qx \quad (p, q \geq 0), \qquad x^{\mathrm{IV}} - (px')' - qx$$

$(p, q \geqslant 0)$, etc. Conditions of the same kind are not difficult to give also for inclusion in $T_{1,n-1}$ $((-1)^{n-k} q_k \geqslant 0,\ k = 0, 1, \dots, n-2)$.

5. A number of fairly general assertions can be obtained for operators represented in the form $Lx \equiv L_0 x + qx$, where $L_0 \in T_0$ and suitable requirements are imposed on the sign of $q(t)$. Let us consider the question of the existence of solutions of (1) satisfying the conditions

$$x(a_i) = x'(a_i) = \dots = x^{(r_i - 1)}(a_i) = 0, \qquad i = 1, 2, \dots, m, \tag{3}$$

r_i even for $i \neq 1, m$ $(m \geqslant 2,\ a \leqslant a_1 < a_2 < \dots < a_m \leqslant b)$.

Theorem 7. Let $r_1 + r_2 + \dots + r_m = n$ and $Lx \equiv L_0 x + qx,\ L_0 \in T_0$. The equation $Lx = 0$ has no nontrivial solutions satisfying conditions (3) in any of the following cases: a) r_m is even and $q(t) \geqslant 0$; b) r_m is odd and $q(t) \leqslant 0$. In particular, if $q(t) \geqslant 0$ $(\leqslant 0)$, then $L \in T_{n-k,k}$ for even (odd) k.

For operators $Lx \equiv x^{(n)} + qx$ the analogous proposition was established by Mikusinski (^{3}).

We shall say that $q(t)$ changes sign on $[a, b]$ in the increasing (decreasing) direction if $q(t)$ changes sign on $[a, b]$, and moreover $(t - t_0) q(t) \geqslant 0$ $(\leqslant 0)$ on $[a, b]$ for some t_0. For what follows the following proposition is useful.

Theorem 8. Let $r_1 + r_2 + \dots + r_m = n + 1$ and $Lx \equiv L_0 x + qx,\ L_0 \in T_0$. The equation $Lx = 0$ has no nontrivial solutions satisfying conditions (3) in any of the following cases: a) r_m is even and $q(t)$ has constant sign or changes sign on $[a, b]$ in the increasing direction; b) r_m is odd and $q(t)$ has constant sign or changes sign on $[a, b]$ in the decreasing direction. In particular, if $q(t)$ has constant sign or changes sign on $[a, b]$ in the increasing (decreasing) direction, then $L \in T_{n-k+1,k}$ for even (odd) k.

Examples of operators of the form $L_0x + qx,\ L_0 \in T_0$ (everywhere $p(t) \geqslant 0$): $x^{(n)} + rx^{(n-1)} - px^{(n-2)} + qx,\ x''' - (px)' + qx,\ x'' + (rx')' + qx,\ x^{\text{IV}} - (px')' + qx,\ x^{\text{IV}} - (px)'' + qx$, etc. Besides these simple examples, which use only the signs of the coefficients and Mammana' s theorem, quantitative criteria of the type of Theorem 5 (see ($^{5-7}$)) may also be used to establish the nonoscillation of L_0; in doing so one should use the nonuniqueness of the representation $Lx \equiv L_1x + qx$.

6. For many applications the question of the monotonicity of the functions $\rho_k(s),\ \mu_k(s)$ is of interest (monotonicity criteria for $\rho_k(s)$ and $\mu_k(s)$ are formulated in a similar way, and therefore in what follows only $\rho_k(s)$ is considered), as well as of the threshold function $\rho(s)$. The expression "$\rho_k(s)$ increases" is used below in the following sense: for any s_1, s_2 such that $a \leqslant s_1 < s_2 \leqslant b$, either $\rho_k(s_1) < \rho_k(s_2) < \infty$, or $\rho_k(s_2) = \infty$; the same applies to $\rho(s)$.

Theorem 9. For a self-adjoint operator of order $n = 2m$, the function $\rho_m(s)$ increases.

Theorem 10. Suppose that for some k,

$$L \in T_{n-k+1,k} \cap T_{n-k,k+1}, \qquad (-1)^n L^* \in T_{k+1,n-k}.$$

Then the function $\rho_k(s)$ increases.

Corollary. Let $Lx \equiv L_0x + qx$, where $L_0 \in T_0$ and $q(t) \geqslant 0$ ($\leqslant 0$). Then for odd (even) k the functions $\rho_k(s)$ increase.

For even (odd) k this assertion is trivial, since in this case, by Theorem 7, $\rho_k(s) \equiv \infty$. Corollary 1 follows from Theorems 7 and 10, if one takes into account that $Lx \equiv L_0x + qx,\ L_0 \in T_0 \to (-1)^n L^*x \equiv (-1)^n L_0^*x + (-1)^{nqx},\ (-1)^n L_0^* \in T_0$.

For the important case $k = 1$, the formulation of Theorem 10 can be modified as follows: if $L \in T^+_{n-1,2},\ (-1)L^* \in T^+_{2,n-1}$, then the function $\rho(s) = \rho_1(s)$ increases. With the aid of Theorem 8 the following proposition is proved.

Theorem 11. Let $Lx \equiv L_0x + qx$, where $L_0 \in T_0$ and $q(t)$ is nonnegative or changes sign on $[a, b]$ in the increasing direction. The inclusion $L \in T$ holds if and only if there exists a function $v(t) \not\equiv 0$ such that

$$v(a) = v'(a) = \cdots = v^{(n-2)}(a) = 0, \qquad v(t) \geq 0, \quad Lv \leq 0$$

$$(a \leq t \leq b).$$

There are known criteria for the inclusion $L \in T$, formulated in terms of the existence of a certain function of two variables; despite their generality, they are, of course, not very effective. Theorem 11, under comparatively general assumptions, makes it possible to restrict oneself to a function of one variable (this is due to the monotonicity of $\rho(s)$). The condition $Lx \equiv L_0x + qx$, $L_0 \in T_0$, in individual cases may be replaced by other requirements ensuring, for example, the inclusion $L \in T_{n-2,2}$, since

$$L \in T_{n-2,2} \to L \in T^{+}_{n-1,2}, \qquad (-1)^n L^* \in T^{+}_{2,n-1}.$$

An example of this kind is furnished by Theorem 2, where the inclusion in $T^{+}_{2,2}$ is ensured by condition 1). Another example is the operator

$$x^V + (p_1x'')'' - (p_2x')' + p_3x,$$

where $p_1, p_2, p_3 \geq 0$; by Theorem 3 it belongs to the class $T_{3,2}$. What has been said also applies to the following assertion.

Theorem 12. Let, under the hypotheses of Theorem 11, $q(t) \geq 0$, $v(b) > 0$. Then for the operator L, for any a_1, b_1 such that $a \leq a_1 < b_1 \leq b$, the Green function $G(t, s)$ of the boundary-value problem

$$x(a_1) = x'(a_1) = \cdots = x^{(n-2)}(a_1) = x(b_1) = 0$$

exists and is negative in the square $a_1 < t, s < b_1$.

7. We give one more assertion, from which, in particular, Theorem 1 of the paper (^{8}) follows directly.

Theorem 13. Let $\rho_k(c) = c_1 < \infty$ for some c, k, and suppose that at least one of the relations

$$c_1 < \min\{\rho_1(c), \dots, \rho_{k-1}(c)\}, \qquad c = \mu_k(c_1) > \max\{\mu_{k+1}(c_1), \dots, \mu_{n-1}(c_1)\}$$

is satisfied. Then the solution of the equation $Lx = 0$ having $(c, n-k;\ c_1, k)$-zeros is unique up to a constant factor and has no zeros inside (c, c_1).

8. For the sake of unity of exposition we have restricted ourselves to linear operators, although in some cases nonlinearities could have been considered. Thus, the assertion analogous to Theorem 7 carries over, without changing the proof, to operators of the form

$$Mx = L_0x + f(t, x),$$

where $L_0 \in T_0$ and

$$xf(t, x) \geq 0 \quad (\leq 0),$$

which corresponds to the case

$$q(t) \geq 0 \quad (\leq 0).$$

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Note: Figure translations are in progress. See original paper for figures.

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