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Abstract

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ON HILBERT'S METHOD FOR SOLVING THE KINETIC BOLTZMANN EQUATION

In recent years the kinetic Boltzmann equation has come to play an exceptionally important role in the study of many contemporary problems of aerodynamics. The investigation of solutions of this equation has been the subject of works by many outstanding scientists. Hilbert ([1]) proposed a method of successive approximations (the usual small-parameter method), which made it possible to reduce the solution of the nonlinear integro-differential Boltzmann equation to the solution of a recurrent system of linear inhomogeneous integral equations. Later Enskog ([2]) proposed another basic method for solving the Boltzmann equation. Enskog' s method is a certain modification of Hilbert' s method. Enskog obtained an explicit solution of the Boltzmann equation. In the zeroth approximation the Euler equations are obtained here, in the first approximation—the Navier-Stokes equations, in the second approximation—the Burnett equations, etc. Unfortunately, up to the present time there is not sufficient clarity concerning the limits of applicability of these methods and their mutual relation.

Some authors arrive at the conclusion that these methods are formally equivalent to one another and lead to identical systems of equations. At the same time it is noted that in Hilbert' s method the number of boundary conditions for the corresponding hydrodynamic equations will, apparently, remain fixed as the number of approximations grows (as $n \to \infty$), whereas in Enskog' s method the number of these boundary conditions may increase as the number of Grad approximations grows ([3]). There are also indications that Hilbert' s method may even lead to erroneous results, which, in particular, was observed already by Boguslavskii ([4]) in the study of standing waves in a gas. From the foregoing it is clear that determining the limits of applicability of Hilbert' s method and Enskog' s method and investigating their mutual relation is of great scientific and practical importance.

The present work is devoted to clarifying the range of applicability of the usual small-parameter method—Hilbert' s method—for solving the Boltzmann equation. It is shown in the work that, in the general form in which this method was applied by Hilbert, it leads to a special structure of the distribution function. In the zeroth approximation, the Euler equations follow from this distribution function. In the first approximation, instead of the Navier-Stokes equations,

inhomogeneous Euler equations are obtained. The situation is not improved in subsequent approximations, which likewise lead to inhomogeneous Euler equations. Only for gas motions with small viscosity and thermal conductivity, in flow regions with small gradients of velocity and temperature, can it be shown that the solution of these inhomogeneous Euler equations will differ little from the corresponding solutions of the Navier-Stokes equations. Thus, the paper shows that the possibilities of the usual small-parameter method are extremely limited.

As is known, the Boltzmann equation can be written in the following simplest form:

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial r} = \frac{s^2}{2m}\iint |(Ve)|\{F'F_1' - FF_1\}\, d\omega\, d\omega_1, \tag{1}$$

where s is the effective diameter, and m is the mass of a molecule.

Let us write equation (1) in abbreviated form:

$$J(F : F) = D(F). \tag{2}$$

Following Hilbert, we represent the distribution function in the form of a power series

$$F(r, v, t) = \frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{(n)} F^{(n)}(r, v, t), \tag{3}$$

where λ is a small parameter.

Substituting (3) into (2) and collecting terms with identical powers of λ, we obtain the recurrent system of integral equations:

$$J\left(F^{(0)}; F^{(0)}\right) = 0,$$

$$J\left(F^{(0)}; F^{(1)}\right) = \frac{1}{2}D\left(F^{(0)}\right),$$

$$J\left(F^{(0)}; F^{(2)}\right) = \frac{1}{2}D\left(F^{(1)}\right) - \frac{1}{2}J\left(F^{(1)}; F^{(1)}\right), \tag{4}$$

……

$$J\left(F^{(0)}; F^{(n)}\right) = \frac{1}{2}D\left(F^{(n-1)}\right) - \frac{1}{2}\sum_{k=1}^{n-1} J\left(F^{(k)}; F^{(n-k)}\right).$$

The first of the equations of this system can be satisfied by the local Maxwellian distribution function

$$F^{(0)}(r,v,t)=\rho_0\left(\frac{m}{2\pi kT_0}\right)^{3/2}e^{-\frac{m(v-U^0)^2}{2kT_0}}, \tag{5}$$

where ρ_0, U^0, and T_0 are certain, as yet unknown, functions of r and t.

For the solvability of the remaining integral equations (4), the following conditions, respectively, must be satisfied:

$$\iiint \psi_i D\left(F^{(0)}\right)\,dv=0,\quad \iiint \psi_i D\left(F^{(1)}\right)\,dv=0,\dots,\quad \iiint \psi_i D\left(F^{(n-1)}\right)\,dv=0, \tag{6}$$

where $\psi_0=1;\ \psi_i=v_i,\ i=1,2,3;\ \psi_4=v^2$.

Introduce the notation:

$$\rho_0(r,t)=\iiint F^{(0)}(r,v,t)\,dv,\qquad \rho_0U^0(r,t)=\iiint vF^{(0)}(r,v,t)\,dv,$$

$$3P_0(r,t)=\iiint (v-U^0)^2F^{(0)}(r,v,t)\,dv. \tag{7}$$

Let us write the first of the conditions (6) in the form

$$\frac{\partial}{\partial t}\left(\iiint \psi_iF^{(0)}\,dv\right)+\frac{\partial}{\partial r_\alpha}\left(\iiint \psi_iv_\alpha F^{(0)}\,dv\right)=0. \tag{8}$$

From expressions (8), using (7), we obtain the system of Euler equations for the functions ρ_0, U^0, and P_0:

$$\frac{\partial\rho_0}{\partial t}+\frac{\partial\rho_0U^0_\alpha}{\partial r_\alpha}=0,$$

$$\frac{\partial U^0_i}{\partial t}+U^0_\alpha\frac{\partial U^0_i}{\partial r_\alpha}+\frac{1}{\rho_0}\frac{\partial P_0}{\partial r_i}=0,\qquad \frac{\partial^{3/2}P_0}{\partial t}+\frac{\partial^{3/2}P_0U^0_\alpha}{\partial r_\alpha}+P_0\ \mathrm{div}\,U^0=0. \tag{9}$$

Thus, in the zeroth approximation the distribution function $F_0(\rho_0,U^0,P_0)$ is completely determined, and the first solvability condition for determining the distribution function in the first approximation $F^{(1)}$ is satisfied.

In order to satisfy the subsequent solvability conditions, Hilbert takes the solution of the inhomogeneous integral equations (4) in the form

$$F^{(n)}(r,v,t) = F^{(n)*}(r,v,t) + F^{(0)}(r,v,t)\sum_{i=0}^{i=4} r_i^{(n)}(r,t)\psi_i. \tag{10}$$

Particular solutions of the inhomogeneous equations $F^{(n)*}(r,v,t)$ are chosen to be orthogonal to the functions ψ_i, i.e.

$$\iiint \psi_i F^{(n)*}(r,v,t)\,dv = 0. \tag{11}$$

The totality of the unknown functions $\chi_i^{(n)}(r,t)$ satisfies an inhomogeneous system of partial differential equations and is determined, generally speaking, uniquely by their initial data.

Let us now determine the form of the distribution function that is obtained in Hilbert' s method. Using (10), we represent the distribution function (3) in the following form:

$$F(r,v,t) = \frac{\widetilde{F}^{(0)} + \lambda F^{(1)*} + \lambda^2 F^{(2)*} + \cdots}{\lambda}, \tag{12}$$

where

$$\widetilde{F}^{(0)}(r,v,t) = F^{(0)}(r,\xi,t)\{1 + \omega_0 + \omega_i\xi_i + \omega_4\xi^2\}. \tag{13}$$

Here the following notation has been adopted:

$$\omega_i(r,t) = \sum_{n=1}^{\infty} \lambda^n \chi_i^{(n)}(r,t).$$

As is seen from (12) and (11), the contribution of the full distribution function to the first moments ρ, U, P is determined only by the function $\widetilde{F}^{(0)}(r,v,t)$. Such a structure of the distribution function is unusual for the kinetic theory of gases. Using this distribution function and Maxwell' s transport equations, we obtain the following system of hydrodynamic equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U_\alpha}{\partial r_\alpha} = 0, \qquad \frac{\partial \rho U_i}{\partial t} + \frac{\partial \rho U_i U_\alpha}{\partial r_\alpha} + \frac{\partial P}{\partial r_i} + \sum_{n=0}^{m} \lambda^n \frac{\partial P_{i\alpha}^{(n)}}{\partial r_\alpha} = 0,$$

$$\frac{\partial^{3/2} P}{\partial t} + \frac{\partial^{3/2} P U_\alpha}{\partial r_\alpha} + P\ \mathrm{div}\, U + \sum_{n=1}^{m} \lambda^n \frac{\partial q_\alpha^{(n)}}{\partial r_\alpha} = 0. \tag{14}$$

In the zeroth approximation ($m = 0$), the system of Euler equations follows naturally from this.

In the first approximation, from (14) we shall have:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U_\alpha}{\partial r_\alpha} = 0, \qquad \frac{\partial \rho U_i}{\partial t} + \frac{\partial \rho U_i U_\alpha}{\partial r_\alpha} + \frac{\partial P}{\partial r_i} + \frac{\partial \sigma_{i\alpha}^{(1)}(U^0)}{\partial r_\alpha} = 0,$$

$$\frac{\partial^{3/2} P}{\partial t} + \frac{\partial^{3/2} P U_\alpha}{\partial r_\alpha} + P \operatorname{div} U + \frac{\partial q_\alpha^{(1)}(U^0)}{\partial r_\alpha} = 0. \tag{15}$$

In the first approximation the functions $\rho(r,t), U(r,t), P(r,t)$ are determined from an inhomogeneous system of Euler equations. The inhomogeneous terms include the components of the viscous stress tensor $\sigma_{i\alpha}^{(1)}$ and the heat-flux vector $q_\alpha^{(1)}$, which are determined through the density, velocity, and pressure in the zeroth approximation (ρ_0, U^0, P_0).

In the general case, the system of equations (14) differs very substantially from the hydrodynamic equations of a viscous fluid. In particular, from (15) it is not possible to obtain such important types of flows of a viscous fluid as Poiseuille flow, flow in the boundary layer, etc. However, for gas flows with small gradients of velocity and temperature, and also for small values of the coefficients of viscosity and thermal conductivity, the solutions of these systems of equations will differ insignificantly.

In the particular case, the usual small-parameter method can lead to the normal structure of the distribution function characteristic of all presently known approximate solutions of the kinetic equation-

including the solutions of Chapman, Enskog, Grad, etc. For this it is sufficient to require of the series (3) that

$$\rho(r,t) = \iiint F(r,v,t)\,dv = \iiint \frac{F^{(0)}(r,v,t)}{\lambda}\,dv,$$

$$\rho U(r,t) = \iiint vF(r,v,t)\,dv = \iiint v\,\frac{F^0(r,v,t)}{\lambda}\,dv, \tag{16}$$

$$3P(r,t) = \iiint (v-U)^2 F\,dv = \iiint (v-U)^2 \frac{F^{(0)}(r,v,t)}{\lambda}\,dv,$$

i.e., that the contribution to the first moments be nonzero only for the distribution function of the zeroth approximation.

The distribution function $F_0(\rho, U, P)$ in the zeroth approximation will also be locally Maxwellian (5), and the aggregate of the functions ρ, U, and P will be

determined by the system of Euler equations (9) from their initial data. In the case under consideration the solvability condition for determining $F^{(1)}$ will also be satisfied. To determine the distribution function in a certain $(n+1)$-st approximation, the solvability conditions may be written in the form:

$$\iiint D\left(F^{(n)}\right)\, dv = 0,$$

$$\iiint v_{iD}\left(F^{(n)}\right)\, dv = \frac{\partial}{\partial r_\alpha} \iiint v_i v_\alpha F^{(n)}\, dv = \frac{\partial P_{i\alpha}^{(n)}}{\partial r_\alpha} = 0, \tag{17}$$

$$\iiint v^2 D\left(F^{(n)}\right)\, dv = \frac{\partial}{\partial r_\alpha} \iiint v^2 v_\alpha F^{(n)}\, dv = \frac{\partial q_\alpha^{(n)}}{\partial r_\alpha} = 0.$$

From these expressions it is clear that already the distribution function of the first approximation $F^{(1)*}$, generally speaking, does not satisfy the solvability conditions for determining the distribution function of the second approximation.

The solvability conditions (17) can be exactly fulfilled only for an ideal gas, and approximately fulfilled for media with vanishingly small viscosity and thermal conductivity at small values of the gradients of velocity and temperature. Thus, even in this particular case the usual small-parameter method does not make it possible to construct, with the proper degree of accuracy, distribution functions of normal structure.

The results obtained in the present work concerning the limited nature of the usual small-parameter method—the Hilbert method—are also confirmed by Boguslavskii' s investigations. In studying standing waves in a gas, using Hilbert' s method, he computed the first approximation (viscous terms) and for the attenuation factor found the expression

$$1 - 2\frac{\mu}{\rho}t. \tag{18}$$

Exact calculations lead to the formula

$$e^{-\frac{2\mu}{\rho}t} = 1 - 2\frac{\mu}{\rho}t + \cdots \tag{19}$$

The limited domain of applicability of the usual small-parameter method is a rather frequent phenomenon. There are possibilities for extending it; some of them will be considered by us in the following paper in connection with another basic method for solving the Boltzmann equations—in connection with Enskog' s method.

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Note: Figure translations are in progress. See original paper for figures.