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1964

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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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ASYMPTOTIC SPLITTING OF THE EQUATIONS OF A CONTROLLED PROCESS UNDER SLOW VARIATION OF THE PARAMETERS OF THE CONTROLLED OBJECT AND OF THE CONTROL SYSTEM

(Presented by Academician A. Yu. Ishlinskii, 15 IV 1964)

1. In the present note we consider the following system of equations, whose study is connected with the solution of the problem of stability of a controlled process in the linear formulation:

$$\frac{dx}{dt} = u(t)x + a(t)z, \quad v(t) = b(t)x,$$

$$z(t) = \int_{-\infty}^t w(t-t', t') v(t') dt', \quad (1)$$

where x is a column matrix composed of the discrepancies of the controlled quantities; z is the control function; $v(t)$ is the input signal of the automatic-control system; $b(t)$ is a row matrix determining the law of formation of the input signal of the control system; $w(t-t', t')$ is the impulse transition function of the automatic-control system; $u(t)$ is a square matrix of order n ; $a(t)$ is a column matrix.

In [1], by asymptotic integration of system (1), under the assumption that the eigenvalues of the matrix u are distinct and sufficiently separated from one another, a stability criterion was obtained that takes into account the variability of the parameters of the controlled object and of the parameters of the automatic-control system. Below we consider the general case, when the eigenvalues of the matrix u may be close and, at individual instants of time, multiple. We reduce the solution of system (1) to the solution of a certain number of mutually independent systems of first-order differential equations. In doing so we use the method of introducing "slow" time proposed by N. M. Krylov and N. N. Bogolyubov [2]. As applied to the system of homogeneous differential equations $dx/dt = u(t)x$, which in our case represents the equations of the "open-loop"

controlled system, this method was used for the asymptotic splitting of the equations into independent subsystems in [3-6].

In what follows, instead of system (1) we shall consider the equivalent system of integro-differential equations

$$\frac{dx}{dt} = u(t)x + a(t) \int_{-\infty}^t w(t-t', t') b(t') x(t') dt'. \quad (2)$$

2. Let the eigenvalues of the matrix $u(t)$ be divided into p groups $\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_{k_\sigma}}$ ($\sigma = 1, 2, \dots, p$), under the condition that on the segment $0 \leq t \leq L$

$$|\lambda_{\sigma_i} - \lambda_{s_j}| > 0 \quad (\sigma \neq s; i = 1, 2, \dots, k_\sigma; j = 1, 2, \dots, k_s). \quad (3)$$

Using Sylvester's formula for the decomposition of matrices into components (see, for example, [7]), in accordance with the indicated partition the matrix u ...

can be represented in the form

$$u = \sum_{\sigma=1}^p K_\sigma \Lambda_\sigma M_\sigma, \quad (4)$$

where $K_\sigma, \Lambda_\sigma, M_\sigma$ are matrices of type $n \times k_\sigma, k_\sigma \times k_\sigma, k_\sigma \times n$, respectively, and

$$M_\sigma K_s = \begin{cases} E_{k_\sigma}, & s = \sigma, \\ 0, & s \neq \sigma \end{cases} \quad (5)$$

(E_{k_σ} is the identity matrix of order k_σ),

$$K_\sigma \Lambda_\sigma M_\sigma = \Delta_{\sigma_1 \dots \sigma_{k_\sigma}}(u) \sum_{i=1}^{k_\sigma} \frac{\lambda_{\sigma_i}}{\Delta_{\sigma_i}(\lambda_{\sigma_i})} \prod_{j \neq i} (u - \lambda_{\sigma_j} E_n),$$

where

$$\Delta_{\alpha_1 \dots \alpha_r}(\lambda) = \frac{\prod_{j=1}^n (\lambda - \lambda_j)}{\prod_{i=1}^r (\lambda - \lambda_{\alpha_i})}.$$

The matrix

$$\sum_{i=1}^{k_\sigma} \frac{\lambda_{\sigma_i}}{\Delta_{\sigma_i}(\lambda_{\sigma_i})} \prod_{j \neq i} (u - \lambda_{\sigma_j} E_n) \equiv \Phi_\sigma(u)$$

is a polynomial of degree $k_\sigma - 1$ in the matrix u . On the segment $[0, L]$ the coefficients of this polynomial are continuous functions of t , including those points at which the eigenvalues belonging to the group σ ($\sigma = 1, 2, \dots, p$) become equal to one another. The indeterminacies that arise at the indicated points are easily resolved.

The rank of the matrix $\Delta_{\sigma_1 \dots \sigma_{k_\sigma}}$ is equal to k_σ . Therefore this matrix can be represented as the product of a matrix K_σ of type $n \times k_\sigma$ and a matrix $M_{0\sigma}$ of type $k_\sigma \times n$ ($\Delta_{\sigma_1 \dots \sigma_{k_\sigma}}(u) = K_\sigma M_{0\sigma}$). After this the matrices M_σ and Λ_σ are determined by the formulas

$$M_\sigma = (M_{0\sigma} K_\sigma)^{-1} M_{0\sigma}, \quad \Lambda_\sigma = M_{0\sigma} \Phi_\sigma K_\sigma$$

(the matrix $M_{0\sigma} K_\sigma$ is nonsingular).

Let us note that expression (4) can also be represented as follows:

$$u = K \Lambda M, \tag{6}$$

where

$$K = (K_1, \dots, K_p), \quad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ \cdot & \cdot \\ 0 & \Lambda_p \end{pmatrix}, \quad M = \begin{pmatrix} M_1 \\ \cdot \\ \cdot \\ M_p \end{pmatrix}.$$

3. Consider the system of integro-differential equations

$$\frac{dx}{dt} = u(\tau)x + \varepsilon a(\tau) \int_{-\infty}^t w(t-t', \tau') b(\tau') x(t') dt' \quad (\tau = \varepsilon t), \tag{7}$$

which for $\varepsilon = 1$ coincides with system (2).

We shall seek the solution of equations (7) in the form

$$x = \sum_{\sigma=1}^p \tilde{K}_\sigma(\tau) \tilde{y}_\sigma(t), \tag{8}$$

where the vector functions \tilde{y}_σ are solutions of the equations

$$\frac{d\tilde{y}_\sigma}{dt} = \tilde{\Lambda}_\sigma(\tau)\tilde{y}_\sigma \quad (\sigma = 1, 2, \dots, p). \quad (9)$$

We represent the matrices $\tilde{K}_\sigma(\tau)$ and $\tilde{\Lambda}_\sigma(\tau)$ in the form of formal expansions

$$\tilde{K}_\sigma = \sum_{k=0}^{\infty} \varepsilon^k K_\sigma^{[k]}, \quad \tilde{\Lambda}_\sigma = \sum_{k=0}^{\infty} \varepsilon^k \Lambda_\sigma^{[k]}. \quad (10)$$

In turn, we seek the solution of equations (9) in the form

$$\tilde{y}_\sigma = \sum_{k=0}^{\infty} \varepsilon^k y_\sigma^{[k]}. \quad (11)$$

Substitute equalities (8) and (9) into equation (7) and equate to zero the sum of all terms containing \tilde{y}_σ . We obtain

$$\left(\varepsilon \frac{d\tilde{K}_\sigma}{d\tau} + \tilde{K}_\sigma \tilde{\Lambda}_\sigma - u\tilde{K}_\sigma \right) \tilde{y}_\sigma - \varepsilon a \int_{-\infty}^t w(t-t', \tau') b(\tau') \tilde{K}_\sigma(\tau') \tilde{y}_\sigma(t') dt' = 0. \quad (12)$$

In the last equality substitute the expansions (9) and (10) and equate to zero the sum of all coefficients at equal powers of ε . We shall have

$$L_\sigma^{[0]} y_\sigma^{[0]} = 0,$$

$$L_\sigma^{[k]} y_\sigma^{[0]} + \sum_{\alpha=1}^k [L_\sigma^{[k-\alpha]} y_\sigma^{[\alpha]} + I_\sigma^{[k-\alpha]} (y_\sigma^{[\alpha-1]})] = 0 \quad (k = 1, 2, \dots), \quad (13)$$

where

$$L_\sigma^{[0]} = K_\sigma^{[0]} \Lambda_\sigma^{[0]} - uK_\sigma^{[0]},$$

$$L_\sigma^{[i]} = \frac{dK_\sigma^{[i-1]}}{d\tau} + K_\sigma^{[i]} \Lambda_\sigma^{[0]} + \dots + K_\sigma^{[0]} \Lambda_\sigma^{[i]} - uK_\sigma^{[i]} \quad (i = 1, 2, \dots), \quad (14)$$

$$I_\sigma^{[i]}(y) = -a(\tau) \int_{-\infty}^t w(t-t', \tau') b(\tau') K_\sigma^{[i]}(\tau') y(t') dt' \quad (i = 0, 1, 2, \dots).$$

Put $K_\sigma^{[0]} \equiv K_\sigma$, $\Lambda_\sigma^{[0]} \equiv \Lambda_\sigma$ (K_σ, Λ_σ are the matrices appearing in expansion (4)). Then, as is not difficult to verify, $L_\sigma^{[0]} \equiv 0$, and in each k -th equality (13) the term containing $y_\sigma^{[k]}$ vanishes.

From equalities (9), (10), (11) there follow the following recurrence formulas for determining $y_\sigma^{[i]}$ ($i = 1, 2, \dots$):

$$y_\sigma^{[i]} = Y_\sigma(t) \int_{t_0}^t Y_\sigma^{-1}(t') [\Lambda_\sigma^{[1]}(\tau') Y_\sigma^{[i-1]}(t') + \dots + \Lambda_\sigma^{[i]}(\tau') Y_\sigma(t')] dt' C = Y_\sigma^{[i]}(t) C \quad (i = 1, 2, \dots), \quad (15)$$

where $Y_\sigma(t)$ is the fundamental matrix of solutions of the equation

$$\frac{dy_\sigma}{dt} = \Lambda_\sigma y_\sigma, \quad (16)$$

and C is a matrix of arbitrary constants.

Taking into account equalities (15), equalities (13) may be brought to the form

$$uK_\sigma^{[k]} = K_\sigma^{[k]} \Lambda_\sigma + K_\sigma \Lambda_\sigma^{[k]} + D_\sigma^{[k-1]} \quad (k = 1, 2, \dots), \quad (17)$$

where

$$D_\sigma^{[k-1]} = \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]} \Lambda_\sigma^{[\alpha]} + \sum_{\alpha=1}^k [L_\sigma^{[k-\alpha]} Y_\sigma^{[\alpha]} + I_\sigma^{[k-\alpha]} (Y_\sigma^{[\alpha-1]})] Y_\sigma^{-1} + \frac{dK_\sigma^{[k-1]}}{d\tau}. \quad (18)$$

If the matrices $Y_\sigma, \Lambda_\sigma^{[1]}, K_\sigma^{[1]}, Y_\sigma^{[1]}, \dots, \Lambda_\sigma^{[k-1]}, K_\sigma^{[k-1]}, Y_\sigma^{[k-1]}$ have already been found, then the matrix $D_\sigma^{[k-1]}$ is a known matrix, and from the k -th equality (17) one can determine $\Lambda_\sigma^{[k]}$ and $K_\sigma^{[k]}$.

Indeed, for compatibility of this equality it is sufficient to require that

$$M_\sigma (K_\sigma \Lambda_\sigma^{[k]} + D_\sigma^{[k-1]}) = 0. \quad (19)$$

Hence we immediately find

$$\Lambda_\sigma^{[k]} = -M_\sigma D_\sigma^{[k-1]}. \quad (20)$$

With the aid of equality (20), the k -th equality (17) can be represented as

$$\Lambda Q_\sigma^{[k]} = Q_\sigma^{[k]} \Lambda_\sigma + M G_\sigma D_\sigma^{[k-1]}, \quad (21)$$

where

$$G_\sigma = E_n - K_\sigma M_\sigma, \quad Q_\sigma^{[k]} = MK_\sigma^{[k]}. \quad (22)$$

We split the matrix $Q_\sigma^{[k]}$, consisting of n rows and k_σ columns, into p blocks so that the first block contains k_1 rows, the second block k_2 rows, and so on:

$$Q_\sigma^{[k]} = \begin{pmatrix} Q_{\sigma 1}^{[k]} \\ \vdots \\ Q_{\sigma p}^{[k]} \end{pmatrix}. \quad (23)$$

Then equality (21) decomposes into p mutually independent matrix equations

$$\Lambda_\sigma Q_{\sigma\sigma}^{[k]} = Q_{\sigma\sigma}^{[k]} \Lambda_\sigma, \quad (24)$$

$$\Lambda_s Q_{\sigma s}^{[k]} = Q_{\sigma s}^{[k]} \Lambda_\sigma + M_s D_\sigma^{[k-1]} \quad (s \neq \sigma). \quad (25)$$

On the basis of equality (24) we may take $Q_{\sigma\sigma}^{[k]} = 0$. Since the matrices Λ_s and Λ_σ for $s \neq \sigma$ have no common eigenvalues, the remaining equalities (25) uniquely determine the matrices $Q_{\sigma s}^{[k]}$ ($s \neq \sigma$). Having thus determined the matrix $Q_\sigma^{[k]}$, it is easy to compute the matrix $K_\sigma^{[k]}$ by the formula

$$K_\sigma^{[k]} = K Q_\sigma^{[k]}. \quad (26)$$

The relations given above make it possible successively to determine the matrices $\Lambda_\sigma^{[1]}, K_\sigma^{[1]}, \Lambda_\sigma^{[2]}, K_\sigma^{[2]}, \dots$, which occur in the formal expansions (10), by means of which the system of integro-differential equations of order n is split into p independent systems of differential equations (9) of orders k_1, k_2, \dots, k_p ($k_1 + k_2 + \dots + k_p = n$). Taking the matrices \tilde{K}_σ and $\tilde{\Lambda}_\sigma$ to be partial sums of the expansions (10), we obtain an approximate splitting of the integro-differential equations. In essence, the integration of equations (7), and hence also of the original system (1), is reduced to the integration of p independent systems of differential equations (16), for, having the matrix $Y_\sigma(t)$ of fundamental solutions of equations (16), one can determine $\tilde{y}_\sigma(t)$ by means of formulas (15) and (11).

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Received
8 IV 1964

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