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Abstract

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ON THE EXISTENCE OF SUBGROUPS IN A FINITE NONSOLVABLE GROUP

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§ 1. The study of the influence of properties of subgroups on the properties of the group is at present a central question in the theory of finite groups. This problem may be formulated more fully as follows. It is known that some part of the subgroups of a finite group G has a group-theoretic property P, while the group itself does not have this property. What can be said about the structure of the group G, if some additional restrictions are imposed on it (for example, nonsolvability)? Questions of this kind are of importance in the proof of D-theorems (^{1,2}). Some results belonging to this circle of ideas may be found in (^{21}). A number of results of (^{21}) are generalized here. In formulating certain theorems, we use the solvability of groups of odd order proved by Thompson and Feit (^{4}).

The proof of many theorems would be considerably simplified if the following conjecture of Herstein (^{5}) were true: a finite group containing a cyclic subgroup coinciding with its normalizer is not simple. However, for our purposes it is enough to prove Herstein' s conjecture in the case when the cyclic subgroup has order equal to twice a prime number (we shall call this assertion the weakened Herstein conjecture).

Only finite groups are considered. The notation and definitions of (^{21}) are used without further explanation. In addition, the following notation and definitions are used. p is always a prime number. A group equal to the direct product of a p-group and a p'-group is called p-decomposable (^{23}) (in this case the p- and p'-groups are considered, by definition, p-decomposable). By a group of type B we mean a nonnilpotent group having the following properties: a) $\tau(G) = 2$; b) one of the Sylow subgroups is invariant in G; c) if $\mathfrak{P}$ is a noninvariant Sylow subgroup of the group G, then all maximal subgroups of $\mathfrak{P}$ belong to the center of the group G (in particular, the subgroup $\mathfrak{P}$ is cyclic).

Let $G = G_1 \times G_2$, where G_1 and G_2 are Hall subgroups of the group G. If G_1 is a pd-subgroup (^{19}) of type S (of type B), then we shall call the group G a group of type S_p (of type B_p). p-Decomposable groups will be regarded as groups of type S_p (of type B_p) by definition.

Attainable (subnormal) subgroups will be regarded as quasinvariant by definition. If a subgroup H is not attainable in a group G, then we shall call it quasinvariant if the least normal divisor of the group G containing H is solvable.

Let

$$C: \quad G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n$$

be a chain of subgroups of the group G such that G_{i+1} is maximal in G_i, $i = 0, 1, \dots, n-1$ (such chains are often called maximal). Denote by $k(C)$ the number of subgroups contained in C that are quasinvariant in G. Since we shall now divide by $k(C)$, we agree to consider only those chains C for which $k(C) > 0$. Put $\bar{v}(C) = n/k(C)$; $\bar{v}(C)$ is called the quasinvariance of the chain C. Put $\bar{v}(G) = \max \bar{v}(C)$, where C runs through all maximal chains with nonzero $k(C)$; $\bar{v}(G)$ will be called the quasinvariance of the group G (cf. (^{7})).

For the definition of the groups $LF(2, p^n)$ and $SL(2, p^n)$, see (^{6}).

§ 2. **Theorem 1.** *Let all solvable subgroups of an unsolvable group G be of type S_2. Then $G = G_1 \times G_2$, where the order of the subgroup G_2 is relatively prime to the number 30 (in particular, G_2 is solvable), and G_1 is isomorphic either to the icosahedral group or to $SL(2,5)$.*

Theorems 2 and 5 of (21) follow from Theorem 1.

Theorem 2. *Let all noninvariant subgroups of an unsolvable group G be of type B. Then G is isomorphic to one of the following three groups:*
$LF(2,5)$, $SL(2,5)$, $LF(2,8)$.

Under the assumption that the weakened Hershtein conjecture is true, the following is proved.

Theorem 3. *Let all solvable subgroups of an unsolvable group G be of type B_2. Then $G = G_1 \times G_2$, where: 1) $G_1 \cong LF(2,5)$ or $SL(2,5)$, and the order of G_2 is relatively prime to 30; 2) $G_1 \cong LF(2,8)$, and the order of G_2 is relatively prime to 42.*

We note that quasinilpotent groups are groups of type B (21). In the fractional-linear group $LF(2,8)$ there is a subgroup of order 18 which is nonnilpotent and not of type S (see the question after Theorem 1 in (21)).

In what follows, π, σ, τ are sets of prime numbers such that $\pi = \sigma \bigcup \tau$, $\sigma \bigcap \tau$ is empty. We shall assume that all prime numbers from π divide the order of the group G. By G_π we denote a π-Hall subgroup of the group G (i.e., a subgroup whose order is equal to the greatest π-divisor of the order of the group G).

Theorem 4. *Let $G_\pi = G_\sigma \times G_\tau$ be a subgroup of type S_2 in the group G, $2 \in \sigma$. If all σ-subgroups of the group G are either nilpotent or of type S, then all π-subgroups of the group G are of type S_2.*

In the proof of Theorem 4, Theorem 1 and the well-known D-theorem of Wielandt (1) are used. Under the assumption that the weakened Hershtein conjecture is true, a theorem analogous to Theorem 4 can be formulated for groups of type B_2. For groups of type S_p, $p > 2$, an analogous result could not be proved. Let a p-Sylow subgroup be invariant in G_σ, $p \in \sigma$. In the formulation of Theorem 4 it is not assumed that in any σ-subgroup of the group G a p-Sylow subgroup is also invariant.

Theorem 5. *Let G be an unsolvable group, $2 \in \pi$. If*

$$sn_\pi(G) < \lambda_\pi(G) + \tau_\pi(G) + 1,$$

then $\pi = \pi(G)$ (21). In particular ((21), Theorem 4), either $G \cong LF(2,5)$, or $G \cong LF(2,11)$.

When counting only πd-subgroups, instead of $r(G)$ (21) we write $r_\pi(G)$.

Theorem 6. *Let G be an unsolvable group, $2 \in \pi$. If*

$$r_\pi(G) < \tau_\pi(G) + 1,$$

then $\pi = \pi(G)$. In particular ((21), Theorem 7), either $G \cong LF(2,5)$, or $G \cong SL(2,5)$.

Theorem 7. *Let G be an unsolvable group with $\bar{v}(G) \leqslant 4$. Then $\bar{v}(G) = 4$ and $\lambda(G) \leqslant 6$. Moreover: 1) if $\lambda(G) = 4$, then $G \cong LF(2,5)$; 2) if $\lambda(G) = 5$, then either $G \cong SL(2,5)$, or $G \cong LF(2,13)$; 3) if $\lambda(G) = 6$, then $G \cong LF(2,p)$. Moreover, $\lambda(p+1) = 3$, $p \not\equiv \pm 1 \pmod 5$.*

Corollary 1. *Let every fourth maximal subgroup of an unsolvable group G be permutable with its conjugates. Then the group G satisfies Theorem 7.*

Corollary 2. *Let every fourth maximal subgroup of an unsolvable group G be quasiinvariant in G. Then the group G satisfies Theorem 7.*

A group from Theorem 7, but without the restriction $p \not\equiv 1 \pmod 5$, will be called a P_3-group. It turns out that these are the only unsolvable groups for which the order of every solvable subgroup H satisfies the relation $\lambda(H) \leqslant 3$.

Theorem 8. *Let G be an unsolvable non-P_3-group. If for every solvable subgroup H with $\lambda(H) = 4$ in the group G there is such a sub-*

proper normal divisor N, such that $HN = G$, then the group G is an extension of a P_3-group by means of a direct product of groups of prime order.

Theorem 9. Let the group G be nonnilpotent and not of type S. If all subgroups of type S are quasi-invariant in the group G, then it is solvable.

In the proof of Theorem 1 the theorem of Frobenius (^{8}), Theorem 16.8.8, is used. In the proof the results of ($^{9-20}, ^{22}$) are used extensively. With the aid of the Thompson—Feit theorem (^{4}) on the solvability of groups of odd order, the following is proved.

Theorem 10. Let all proper subgroups of the *pd*-group G satisfy the condition C_π (^{8}). If the group G does not satisfy the condition C_π, then it is an extension of its Frattini subgroup by means of a nonsolvable simple group (which, obviously, does not satisfy the condition C_π).

In the proof of Theorem 10 one also uses the fact that if all factors of a normal series of the group G satisfy the condition C_π, then G also satisfies this condition (^{8}).

If M and N are two such sets of subgroups of the group G that every subgroup from M is permutable with every subgroup from N, then we shall say that these sets are permutable. If $M \neq N$, we write $MN = NM$. If $M = N$, then we shall say that the set M is commutative. Denote by Γ_i the set of all i-th maximal subgroups of the group G.

Theorem 11. If G is a nonsolvable group, then the following properties are equivalent:
1) $\Gamma_1\Gamma_4 = \Gamma_4\Gamma_1$; 2) $\Gamma_2\Gamma_4 = \Gamma_4\Gamma_2$; 3) $\Gamma_3\Gamma_4 = \Gamma_4\Gamma_1$; 4) $\bar{v}(G) = 4$.

Let

$$H = F_1 \times \dots \times F_m \times S_1 \times \dots \times S_n \times N,$$

where F_i, S_j, N are groups of pairwise relatively prime orders; moreover F_i, $i = 1, \dots, m$, are Frobenius groups, S_j, $j = 1, \dots, n$, are groups of type S, and the group N is nilpotent (zero values of the numbers m, n are allowed). Then we shall call G a group of type C.

Theorem 12. Let all solvable subgroups of a nonsolvable group G of type C and not of type C be of type C. Then $G = G_1 \times G_2$, where G_1 and G_2 are Hall subgroups of the group G, the subgroup G_2 is of type C, and the subgroup G_1 is of one of the following forms: 1) the fractional-linear group $LF(2, 2^\alpha)$, $\alpha > 1$; 2) the simple Suzuki group $G(2^{2n+1})$, $n > 0$ (^{9}); 3) the special linear group $LS(2, 5)$.

Theorem 13. Let all nonnilpotent solvable subgroups of a nonsolvable group G be of type B. Then the group G is isomorphic to one of the groups $LF(2, 5)$, $SL(2, 5)$, $LF(2, 8)$.

From Theorem 7 follows a result of Janko (^{25}).

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REFERENCES

1. H. Wielandt, Math. Zs., **60**, 407 (1954).

2. H. Wielandt, Math. Zs., **71**, 461 (1959).
3. O. H. Kegel, Arch. Math., **12**, 90 (1961).
4. W. Feit, J. G. Thompson, Proc. Nat. Acad. Sci. U. S. A., **48**, 6, 968 (1962).
5. D. Gorenstein, I. N. Herstein, Am. J. Math., **83**, 71 (1961).
6. L. E. Dickson, *Linear Groups with an Exposition of Galois Field Theory*, Leipzig, 1901.
7. W. E. Deskins, Illinois J. Math., **5**, 306 (1961).
8. P. Hall, Proc. London Math. Soc. (3), **6**, 286 (1956).
9. M. Suzuki, Ann. Math., **75**, 1, 105 (1962).
10. I. Schur, J. reine u. angew. Math., **132**, 85 (1907).
11. H. Wielandt, Abh. Math. Sem. Hamburg, **22**, 215 (1958).
12. J. C. Thompson, Proc. Nat. Acad. Sci. U. S. A., **45**, 4, 578 (1959).
13. H. Wielandt, J. reine u. angew. Math., **182**, 180 (1940).
14. R. Brauer, M. Suzuki, Proc. Nat. Acad. Sci. U. S. A., **45**, 12, 1757 (1959).
15. W. Feit, Am. J. Math., **82**, 281 (1960); **84**, 201 (1962).
16. J. G. Thompson, Math. Zs., **72**, 458 (1960).
17. R. Brauer, M. Suzuki, G. E. Wall, Illinois J. Math., **2**, 718 (1958).
18. B. Huppert, Math. Zs., **60**, 409 (1954).
19. S. A. Chunikhin, Mat. sborn., **25** (67), 321 (1949).
20. Yu. A. Gol' fand, DAN, **60**, No. 8, 1313 (1948).
21. Ya. G. Berkovich, DAN, **151**, No. 5, 1007 (1963).
22. S. A. Chunikhin, DAN, **118**, No. 4, 654 (1958).
23. I. K. Chunikhin, S. A. Chunikhin, Mat. sborn., **15** (57), 325 (1944).

24. M. Hall, *Theory of Groups*, Moscow, 1962.

25. Z. Janko, Math. Zs., **82**, 82 (1963).

Note: Figure translations are in progress. See original paper for figures.

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