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Abstract

Full Text

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G. N. SYTAYA

ON THE LIMITING DISTRIBUTION OF A CERTAIN CLASS OF FUNCTIONALS OF A SEQUENCE OF SUMS OF INDEPENDENT RANDOM VARIABLES

(Presented by Academician A. N. Kolmogorov, 7 IV 1964)

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of independent identically distributed random variables with $M\xi_n = 0,\ D\xi_n = 1.$ Denote

$$S_{nk} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \xi_i; \qquad S_{n0} = 0.$$

Consider a sequence of nonnegative measurable functions $\Phi_n(x, y)$, defined for $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$. We shall seek conditions sufficient for the existence of a limiting distribution of the quantities

$$\eta_n = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \Phi_n(S_{nj}, S_{nk}).$$

If $\Phi(x, y)$ is a continuous function, then from M. Donsker's theorem ([1]) it follows that the limiting distribution of the quantities

$$\eta_n = \frac{1}{n^2} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \Phi(S_{nj}, S_{nk})$$

exists and coincides with the distribution of the functional

$$\int_0^1 \int_0^1 \Phi(w(t), w(s))\, dt\, ds$$

$(w(t)$ is the Brownian motion process).

For arbitrary nonnegative measurable functions $\Phi_n(x)$ such that

$$\lim_{n\to\infty} \sup \Phi_n(x) = 0,$$

the problem was considered by A. V. Skorokhod in the paper ([2]), in which it is established that, for the existence of a limiting distribution of the quantities

$$\eta_n = \sum_{j=0}^{n-1} \Phi_n(S_{nj}),$$

it is sufficient that: 1) there exist a square-integrable density of the random variables ξ_j, and 2) the sequence of functions

$$u_n(x) = 2n \int_0^x \Phi_n(z)\, dz$$

converge almost everywhere to some limit $u(x)$. In this case the limiting distribution of the quantities η_n coincides with the distribution of the random variable

$$\int_0^{w(1)} u(x)\, dx - \int_0^1 u(w(s))\, dw(s).$$

This result was generalized to the case of sign-changing functions by N. P. Slobodenyuk ([3]).

On the basis of the results of article ([2]), one can prove the validity of the following assertion:

Theorem. *Suppose that the following hold:*

1. *The ξ_j have a square-integrable density $p(x)$,*

$$\int |x|^3 p(x)\, dx < \infty.$$

2.

$$\lim_{n\to\infty} \sup_{x,y} n^{3/2} \Phi_n(x, y) = 0.$$

3. *For each N, the quantities $nu_n(x, y)$, $nu_n^*(x, y)$, where*

$$u_n(x, y) = 2n \int_0^x \Phi_n(\alpha, y)\, d\alpha; \qquad u_n^*(x, y) = 2n \int_0^y \Phi_n(x, \beta)\, d\beta,$$

are uniformly bounded in the aggregate on the set $|x| \le N$; $|y| \le N$.

4. *The sequence*

$$q_n(x,y) = 4n^2 \int_0^x \int_0^y \Phi_n(\alpha,\beta)\, d\alpha\, d\beta$$

converges almost everywhere to the function $q(x,y)$.

Then the limiting distribution of η_n *exists and coincides with the distribution of the random variable*

$$\int_0^{w(1)} \int_0^{w(1)} q(x,y)\, dx\, dy - \int_0^1 \int_0^{w(1)} q(x,w(s))\, dx\, dw(s) -$$

$$- \int_0^{w(1)} \int_0^1 q(w(t),y)\, dw(t)\, dy + \int_0^1 \int_0^1 q(w(t),w(s))\, dw(t)\, dw(s), \qquad (1)$$

where

$$\int_0^1 \int_0^{w(1)} q(x,w(s))\, dx\, dw(s) = p \lim_{n\to\infty} \sum_{k=0}^{n-1} \int_0^{w(1)} q(x,w(t_k))\, dx\, \Delta w(t_k),$$

$$\int_0^1 \int_0^1 q(w(t),w(s))\, dw(t)\, dw(s) = p \lim_{n\to\infty} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} q(w(t_j),w(t_k))\, \Delta w(t_j) \Delta w(t_k).$$

The proof of the theorem is based on the following lemmas.

Lemma 1. *Let* $\Phi_n(x,y) = 0$ *for* $|x| > c$ *or* $|y| > c$. *Then, if conditions 1 and 2 of the theorem are satisfied and the functions*

$$n\bar{u}_n(x,y) = 2n^2 \int_{-\infty}^x \Phi_n(\alpha,y)\, d\alpha$$

are bounded in the aggregate, the quantities η_n *have a limiting distribution if and only if there exists a limiting distribution of the quantities*

$$\sum_{k=0}^{n-1} \int_0^{S_{nn}} \bar{u}_n(x,S_{nk})\, dx - \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \bar{u}_n(S_{nj}S_{nk}) \frac{\xi_{j+1}}{\sqrt{n}}. \qquad (2)$$

and the two distributions coincide.

Lemma 2. *If the conditions of Lemma 1 are satisfied and the quantities*

$$n\bar{u}_n^*(x,y) = 2n^2 \int_{-\infty}^{y} \Phi_n(x,\beta)\,d\beta$$

are bounded in the aggregate, then the limiting distribution of (2), and consequently also of η_n, exist simultaneously, and if (2) exists, then it coincides

with limiting distribution

$$\int_0^{S_{nn}} \int_0^{S_{nn}} \overline{q}_n(x,y)\,dx\,dy - \sum_{k=0}^{n-1} \int_0^{S_{nn}} \overline{q}_n(x,S_{nk})\,dx\,\frac{\xi_{k+1}}{\sqrt{n}} - \sum_{j=0}^{n-1} \int_0^{S_{nn}} \overline{q}_n(S_{nj},y)\,dy\,\frac{\xi_{j+1}}{\sqrt{n}} + \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \overline{q}_n(S_{nj},S_{nk})\frac{\xi_{j+1}}{\sqrt{n}} \tag{3}$$

where

$$\overline{q}_n(x,y) = 4n^2 \int_{-\infty}^{x} \int_{-\infty}^{y} \Phi_n(\alpha,\beta)\,d\alpha\,d\beta.$$

Lemma 3. If the conditions of Lemmas 1 and 2 are satisfied and the sequence $\overline{q}_n(x,y)$ converges almost everywhere to a nondecreasing function $\overline{q}(x,y)$, then the limiting distribution of the quantity (3) exists and coincides with the distribution of the random variable

$$\int_0^{w(1)} \int_0^{w(1)} \overline{q}(x,y)\,dx\,dy - \int_0^1 \int_0^{w(1)} \overline{q}(x,w(s))\,dx\,dw(s)$$

$$- \int_0^{w(1)} \int_0^1 \overline{q}(w(t),y)\,dw(t)\,dy + \int_0^1 \int_0^1 \overline{q}(w(t),w(s))\,dw(t)\,dw(s).$$

Example. Suppose that, for $\xi_1, \xi_2, \dots, \xi_n, \dots$, condition 1 of the theorem is satisfied. Consider the function

$$\Phi_n(x,y) \begin{cases} \dfrac{1}{n^2\varepsilon_n}, & \text{when } |x-y| < \varepsilon_n, \\ 0, & \text{when } |x-y| \ge \varepsilon_n. \end{cases}$$

It is not difficult to verify that if ε_n is chosen equal to $n^{-1/2+\delta}$ $(0 < \delta < 1/2)$, then conditions 2–4 of the theorem are satisfied, and the sequence $q_n(x,y)$ converges to the value

$$q(x,y) = \begin{cases} 0, & \text{when } x \cdot y \le 0, \\ 8\min(|x|,|y|), & \text{when } x \cdot y > 0. \end{cases}$$

Consequently, there exists a limiting distribution of the functional η_n, coinciding with the distribution of the random variable (1). In the present case formula (1), with the aid of K. Ito' s theorem[4], can be transformed into the form

$$4\left\{\int_0^1 |w(t)-w(1)|\,dt - \int_0^1 |w(t)|\,dt + w(1)\int_0^1 \operatorname{sgn} w(t)\,dt\right\}$$

$$-\frac{8}{3}|w(1)|^3 + \int_0^1\int_0^1 q(w(t),w(s))\,dw(t)\,dw(s).$$

In conclusion I express my sincere gratitude to A. V. Skorokhod, who posed the problem and guided its solution.

Institute of Mathematics
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.