B. S. MITYAGIN

1. The well-known theorem of Bernstein asserts:

1964

SovietRxiv

Abstract

Full Text

B. S. MITYAGIN

ON THE ABSOLUTE CONVERGENCE OF A SERIES OF FOURIER COEFFICIENTS

(Presented by Academician A. N. Kolmogorov on II 8, 1964)

1. The well-known theorem of Bernstein asserts:

If a periodic function $f(t)$ on $[0, 2\pi]$ belongs to the class C^α, $\alpha > 1/2$, i.e., satisfies the Hölder condition with exponent $\alpha > 1/2$, then the series of Fourier coefficients with respect to the trigonometric system converges absolutely:

$$\sum_{-\infty}^{\infty} \left| \int_0^{2\pi} f(t) e^{-imt}\, dt \right| < \infty;$$

the exponent $1/2$ is sharp, i.e., there exists a function $f_0 \in C^{1/2}$ such that

$$\sum_{-\infty}^{\infty} \left| \int_0^{2\pi} f_0(t) e^{-imt}\, dt \right| = \infty.$$

This result was strengthened and generalized by many authors (see (^{1}), Ch. 9) in various directions. For the Haar system $\chi_n(t)$ (see (^{4}), p. 141) Ciesielski and Musielak proved (^{2}) the following fact:

Lemma of Ciesielski–Musielak. If a function

$$f(t) \in L_1^\alpha[0, 1], \quad \alpha > 1/2,$$

i.e., $f \in L_1(0, 1)$ and

$$\omega_1(\delta; f) = \sup_{|h| \le \delta} \int_0^1 |f(t+h) - f(t)|\, dt = O(\delta^\alpha), \quad \delta \downarrow 0,$$

then

$$\sum_1^{\infty} \left| \int_0^1 f(t) \chi_n(t)\, dt \right| < \infty.$$

P. L. Ul' yanov established (^{3}), §8, that in this case as well the Hölder exponent 1/2 is limiting and sharp; namely, in $L_1^{1/2}$ there exists a (even bounded) function $f_0(t)$ such that

$$\sum_1^\infty |a_n(f_0)| = \infty, \quad \text{where} \quad a_n(f_0) = \int_0^1 f_0(t)\chi_n(t)\,dt.$$

The result of P. L. Ul' yanov was strengthened by B. I. Golubov, who showed* the existence of such a function $f_0(t)$ in the class $C^{1/2}$. The main purpose of the present note is to establish the general fact that the Hölder exponent 1/2 in a theorem of Bernstein type cannot be improved for any orthonormal system. The following is true:

Theorem 1. Let $\Psi = \{\psi_n\}$ be an arbitrary complete orthonormal system on $[0, 2\pi]$; there exists a function $\varphi(t) \in C^{1/2}[0, 2\pi]$ such that the series of its Fourier coefficients with respect to the system Ψ does not converge absolutely, i.e.,

$$\sum_1^\infty \left| \int_0^{2\pi} \varphi(t)\psi_n(t)\,dt \right| = \infty.$$

Thus, in particular, a new proof is given of the second part of Bernstein' s theorem. Multidimensional analogues of this assertion have also been found.

2. Here is the scheme of the proof of Theorem 1:

$$\begin{tikzcd}
& C^{1/2} \arrow[r, "I"] & l_1 \arrow[d, "J", "2"'] \\
L_\infty \arrow[ur, "\Lambda^{1/2}", "3"'] \arrow[rr, "\Lambda^{1/2}", "4"'] & & L_2
\end{tikzcd}$$

What is meant by each of the operators indicated above the arrows is explained below; the continuity of these operators and their additional properties are established in the lemmas whose numbers are written under the arrows.

Suppose that every function $\varphi(t) \in C^{1/2}$ has an absolutely convergent Fourier coefficient series with respect to the system Ψ. Then define the operator

$$I : C^{1/2} \to l_1, \quad \text{setting} \quad I\varphi = \{\xi_n\}, \quad \text{where} \quad \xi_n = \int_0^{2\pi} \varphi(t)\psi_n(t)\,dt.$$

---

* The reference to this still unpublished result of B. I. Golubov is made here with his kind permission.

Lemma 1. *The operator* $I : C^{1/2} \to l_1$ *is continuous if* $\{\xi_n\} \in l_1$ *for all* $\varphi \in C^{1/2}$.

This is a special case of Gelfand' s theorem (see (^{4}), pp. 432-433). The operator J is constructed by the formulas $\xi = \{\xi_n\} \in l_1,\ J\xi = \varphi,\ \varphi(t) = \sum \xi_n \psi_n(t) \in L_2$. Since $\sum |\xi_n|^2 \leq \left(\sum |\xi_n|\right)^2$, we have $\|J\xi\|_2 \leq \|\xi\|_1$, and the operator $J : l_1 \to L_2$ is continuous. But it is important for us to know more about J; namely, the following is true (^{5}).

Lemma 2. *The operator* $J : l_1 \to L_2$ *is unconditionally summing.*

Recall that an operator $T : X \to Y$ from one Banach space to another is called **unconditionally summing**, or a Grothendieck operator, if one of the following two equivalent (^{6}) conditions is satisfied:

1°. For every unconditionally convergent series $\sum x_n$ in X, the series $\sum Tx_n$ converges absolutely in Y, i.e. $\sum \|Tx_n\|_Y < \infty$.

2°. For some constant M,

$$\sum_{n=1}^{N} \|Tx_n\|_Y \leq M \sup_{|\xi_n| \leq 1} \left\| \sum_{1}^{N} \xi_n x_n \right\|_X$$

for any $x_n \in X,\ n = 1, 2, \ldots, N;\ N = 1, 2, \ldots$.

Finally, Λ^γ is the multiplier operator with respect to the trigonometric system:

$$\Lambda^\gamma(1) = 1, \qquad \Lambda^\gamma(e^{int}) = |n|^{-\gamma} e^{int}, \quad n \neq 0.$$

The Hardy-Littlewood theorem ((^{7}), p. 223, Theorem 9.81) can be reformulated as follows:

Lemma 3. *The operator* $\Lambda^\gamma : L_\infty \to C^\gamma$ *is continuous if* $0 < \gamma < 1$. Note that the same is also true in the metrics $L_p,\ 1 \leq p < \infty$, i.e. $\Lambda^\gamma : L_p \to L_p^\gamma$ is continuous.

3. Since Ψ is a complete system, the product $JI : C^{1/2} \to L_2$ is the identity embedding operator. By Lemmas 1-3, the operator $\Lambda^{1/2} : L_\infty \to L_2$ has been represented as the product of several continuous operators, one of which, J, is unconditionally summing. Therefore the operator $\Lambda^{1/2} : L_\infty \to L_2$ would have to be unconditionally summing. However, this is false.

Lemma 4. *The operator* $\Lambda^{1/2} : L_\infty \to L_2$ *is not unconditionally summing.*

Indeed, let us show that condition 2° is not satisfied. For any N, put

$$x_n(t) = \chi_{\Delta_n}(t), \qquad \Delta_n = \left[\frac{n}{N}2\pi, \frac{n+1}{N}2\pi\right], \quad n = 0, 1, \ldots,$$

$$\ldots, N-1,$$

and let χ_E be the characteristic function of the set E. Since the functions $x_n(t)$ are obtained from one another by translation, the moduli of their Fourier coefficients with respect to the trigonometric system coincide, and, as is easy to compute, for $x_0(t)$, $|\xi_k| = |\sin k\varepsilon/k|$, where $\varepsilon = \pi/N$. Therefore

$$\sum_1^N \|\Lambda^{1/2}x_n\|_2 = N\|\Lambda^{1/2}x_0\|_2 \geq N\left(\sum_1^\infty \frac{\sin^2 k\varepsilon}{k^2}\frac{1}{k}\right)^{1/2} \geq$$

$$\geq N\left(\sum_1^{N/2} \frac{4}{\pi^2}\frac{k^2\pi^2}{N^2k^2}\frac{1}{k}\right)^{1/2} \geq \left(\log\frac{N}{2}\right)^{1/2}.$$

Here we used the fact that $\sin x \geq \frac{2}{\pi}x$ if $0 \leq x \leq \frac{\pi}{2}$. Thus condition 2° is not satisfied, and Lemma 4 is proved.

The contradiction obtained shows that the assumption of absolute convergence of the series of Fourier coefficients with respect to the system $\Psi = \{\psi_n\}$ for all functions $\varphi \in C^{1/2}$ is not satisfied. Theorem 1 is proved.

4. A very essential point in the proof of Theorem 1 was the theorem of Mazur–Pelczyński–Schlenk (Lemma 2); it is precisely this theorem that gives the above scheme a general character. Thus, if one has to solve the analogous question for some space E of functions of one or several variables, it suffices to choose an operator Λ (and a space X) such that, on the one hand, $\Lambda : X \to E$ is continuous, and on the other hand $\Lambda : X \to L_2$ is not unconditionally summing. Then consideration of the commutative diagram

$$\begin{tikzcd} & E \arrow[r, "I"] & \ell_1 \arrow[d, "J"] \\ X \arrow[ur, "\Lambda"] \arrow[rr, "\Lambda"] & & L_2 \end{tikzcd}$$

as above will lead to a contradiction. Thus the problem is reduced to a successful choice of the operator Λ. Before giving one of the many possible multidimensional analogues of Theorem 1, we shall state some propositions on the connection between constructive and differential properties of functions of several variables. These facts perhaps also have independent interest.

For a set of positive numbers $\alpha = (\alpha_1, \dots, \alpha_m)$ we shall denote by C^α the space of m-periodic functions $f(t_1, \dots, t_m)$ such that, if $\alpha_j = p_j + \gamma_j$, p_j is an integer, $0 < \gamma_j \leqslant 1$, then for every $j = 1, 2, \dots, m$

$$\left|\frac{\partial^{p_j} f(t+u^j)}{\partial t_j^{p_j}} - \frac{\partial^{p_j} f(t)}{\partial t_j^{p_j}}\right| \leqslant M_f|u_j|^{\gamma_j} \quad \text{for all } u = (u_1, \dots, u_m);$$

here

$$t+u^j=(t_1,\dots,t_{j-1},\,t_j+u_j,\,t_{j+1},\dots,t_m).$$

We set

$$\frac{1}{\alpha_0}=\sum_1^m\frac{1}{\alpha_j}.$$

Put

$$P_\alpha(n)=\left(1+\sum_1^m|n_j|^{2\alpha_j}\right)^{-1/2},\qquad n=(n_1,\dots,n_m),$$

and let $E^{(p)}_{R,P_\alpha}f$ be the best approximation in the metric L_p, $1\leqslant p\leqslant\infty$, of the function f by linear combinations of those exponentials $\exp\{i(n,t)\}$, $(n,t)=\sum_1^m n_{jt}j$, for indices n for which $P_\alpha(n)\leqslant R$.

Theorem 2. If $f\in C^\alpha(L_p^\alpha)$, then

$$E^{(p)}_{R,P_\alpha}f\leqslant CM_fR^{-1}.$$

Conversely, if

$$E^{(p)}_{R,P_\alpha}f\leqslant N_fR^{-1},$$

then $f\in L_p^\alpha$ and

$$M_f\leqslant C(\alpha,p)N_f,$$

except for the cases when α_0 is an integer and $p=1,\infty$.

The proof is carried out according to the schemes of article (8), where, in a less exact form (Theorem 6, p. 411), an analogous fact was obtained for the case of even $\alpha_j,\ j=1,\dots,m$.

Theorem 3. If $\alpha_0>1/2$, then every function $f\in C^\alpha$ has an absolutely convergent Fourier series with respect to the trigonometric system. If, however, $\alpha_0\leqslant 1/2$, then whatever complete orthonormal system

$$\Psi=\{\psi_n(t_1,\dots,t_m)\},\qquad 0\leqslant t_j\leqslant 2\pi,$$

there exists a function $\varphi\in C^\alpha$ such that

$$\sum \left| \int \varphi \psi_n \, dm^t \right| = \infty.$$

The first part is verified* according to the scheme of the proof of Bernstein' s theorem (for example, (10), p. 594) using Parseval' s equality and the estimates following from Theorem 2,

$$E^{(2)}_{R,P_\alpha} f \leqslant E^{(\infty)}_{R,P_\alpha} f \leqslant C M_f R^{-1}.$$

The second part is proved according to the scheme of the second diagram 2, where $X = L_\infty$ and $\Lambda = \Lambda^\alpha$ is the multiplier operator $\{\lambda_n\}$, $\lambda_n = P_\alpha(n)$.

Lemma 5. *If $f \in L_\infty$ and $\alpha_0 < 1$, then*

$$E^{(\infty)}_{R,P_\alpha}(\Lambda^\alpha f) \leq C \|f\|_\infty R^{-1}.$$

The proof is carried out according to the scheme of article (8) (cf. Theorem 2 on p. 404).

Lemma 3a. *The operator $\Lambda^\alpha : L_\infty \to C^\alpha$ is continuous.*

This is a consequence of Lemma 5 and the second part of Theorem 2.

Thus, by Lemmas 1, 2, 3a, the operator $\Lambda^\alpha : L_\infty \to L_2$ is unconditionally summing.

Lemma 4a. *The operator $\Lambda^\alpha : L_\infty \to L_2$ is not unconditionally summing if $\alpha_0 \leq \frac{1}{2}$.*

For the proof, one considers the characteristic functions χ_{Δ_n} of equal cubes obtained by partitioning the sides of the cube $0 \leq t_j \leq 2\pi$ into N equal parts. Simple calculations show that

$$\left\| \sum_1^{N^m} \Lambda^\alpha \chi_{\Delta_n} \right\|_2 = N^m \left\| \Lambda^\alpha \chi_{\Delta_1} \right\|_2 \geq C \left(\int_{1 \leq x_j \leq N} \cdots \int \frac{dx_1 \cdots dx_m}{1 + x_1^{2\alpha_1} + \cdots + x_m^{2\alpha_m}} \right)^{1/2},$$

and the divergence of such an integral, taken over the whole space R^m, for $\alpha_0 \leq \frac{1}{2}$ (see (10), example 86, item 676, p. 397; for $m = 2$, example 26, item 617, p. 225) ensures the failure of condition 2^0. The contradiction to which Lemma 4a leads proves Theorem 3.

---

* This result is not new; many sufficient conditions for absolute convergence of multidimensional Fourier series were given by M. F. Timan (12). Let us also note that absolute convergence of expansions in eigenfunctions of elliptic operators for functions from C^α, $\alpha = (\beta, \beta, \dots, \beta)$, β an integer and $\beta > m/2$, was established by V. A. Il' in (12).

6. We shall make a few concluding remarks. The structure of Bernstein' s theorem is as follows: given a homogeneous space (11) of functions X on the circle (on the m-dimensional torus), $L^1 \supset X \supset C$ (or L^∞), and some complete orthonormal system Ψ; one seeks such a Hölder exponent $\alpha = \alpha(X, \Psi)$ that all functions of the classes $X^{\alpha+\varepsilon}$, $\varepsilon > 0$, have an absolutely convergent series of Fourier coefficients with respect to the system Ψ, while in the class X^α this is false ($X^\gamma = \{f \in X : \|f(t+h) - f(t)\|_X \leq M_f |h|^\gamma\}$). For the trigonometric system $\Phi = \{e^{int}\}$ the exponent
$$\alpha(L_p, \Phi) = \begin{cases} \frac{1}{2}, & 2 \leq p \leq \infty, \\ 1/p, & 1 \leq p \leq 2. \end{cases}$$
Although for the trigonometric system $\alpha(L_p, \Phi) = 1/p > \frac{1}{2}$ for $p > 2$, the estimate of Theorem 1, $\alpha(L_p, \Psi) \geq \frac{1}{2}$, in the class of all complete orthonormal systems, as the Ciesielski–Musielak lemma shows, cannot be improved.

Finally, let us note that the following is true.

Proposition. *If* $\sup |\lambda_k| \cdot k^\gamma < \infty$, $\gamma > \frac{1}{2}$, *then the multiplier operator* $\Lambda = \{\lambda_k\}$ *with respect to the trigonometric system* $\Lambda : L_1 \to L_2$ *is unconditionally summing. In order that the operator* $\Lambda : L_\infty \to L_2$ *be unconditionally summing, it is necessary and sufficient that the condition* $\sum |\lambda_k|^2 < \infty$ *hold.*

Voronezh State University

Received
8 II 1964

REFERENCES

1. N. K. Bari, *Trigonometric Series*, Moscow, 1961.
2. Z. Ciesielski, J. Musielak, Coll. Math., 7, No. 1 (1959).
3. P. L. Ul' yanov, Russian Math. Surveys, 19, No. 1 (115), § 8 (1964).
4. S. Kaczmarz, H. Steinhaus, *Theorie der Orthogonalreihen*, Moscow, 1958.
5. A. Pełczyński, W. Shlenk, Coll. Math., 10, No. 2, 313 (1963).
6. M. S. MacPhail, Bull. AMS, 53, 121 (1947).
7. A. Zygmund, *Trigonometric Series*, 1939.
8. B. S. Mityagin, Mat. sbornik, 58, 4, 397 (1962).

9. I. P. Natanson, *Constructive Function Theory*, 1949.

10. G. M. Fichtenholz, *A Course of Differential and Integral Calculus*, 3, Moscow, 1960.

11. G. E. Shilov, *Mathematical Analysis*, Moscow, 1961, p. 313.

12. M. F. Timan, Doctoral dissertation, Dnepropetrovsk, 1963; A. V. Il' in, Russian Math. Surveys, No. 1 (79), 87 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.