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Abstract

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INVESTIGATION OF THE STABILITY OF THE BUBNOV-GALERKIN METHOD FOR NONSTATIONARY PROBLEMS

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In the present note we use, without reservation, the notation, terminology, and results of the monograph of S. G. Mikhlin ([1]). The paper studies the stability of the Bubnov-Galerkin method for equations of parabolic and hyperbolic types, as well as for equations with operator coefficients containing the second and first derivatives with respect to time. Here the "coordinate system" is not chosen arbitrarily.

S. G. Mikhlin ([2,3]) considered the equation $Au = f$ in a Hilbert space H with a positive definite self-adjoint operator A and proved that the Ritz method for this equation is stable with respect to small changes of the coefficients and the free terms of the Ritz system, if the coordinate system is strongly minimal ([4]) in the space H_A. These works served as the starting point for our investigation. Let us also note the article ([5]), in which the stability of the Galerkin-Petrov method for stationary problems is studied.

Consider the problem

$$\mathcal{L}u \equiv A\frac{d^2u(t)}{dt^2} + B\frac{du(t)}{dt} + Cu(t) = f(t), \tag{1}$$

$$u(t)\Big|_{t=0} = \varphi, \qquad \frac{du(t)}{dt}\Big|_{t=0} = \psi, \tag{2}$$

where $f(t)$ is a given function and $u(t)$ the unknown function with values in H; A, B, C are self-adjoint, time-independent operators acting in a certain Hilbert space H, such that they are nonnegative and at least two of them are positive definite in H, while φ and ψ are given elements of the space H.

We shall assume that the domains of definition $D(A), D(B)$, and $D(C)$ intersect in a set $\mathfrak{M}$, everywhere dense simultaneously in all three spaces H_A, H_B, and H_C. On the set $\mathfrak{M}$ we define a new scalar product by putting

$$[u,v]_0 = (Au,v) + (Bu,v) + (Cu,v); \qquad u \in \mathfrak{M},\ v \in \mathfrak{M}.$$

By virtue of this definition, $\mathfrak{M}$ becomes a new Hilbert space, which we shall denote by H_0. The norm in H_0 will be denoted by

$$\|u\|_0^2 = \|u\|_A^2 + \|u\|_B^2 + \|u\|_C^2; \qquad u \in \mathfrak{M}.$$

If it turns out that the space H_0 is incomplete, we complete it in the usual way. Obviously, $\mathfrak{M}$ will be dense in H_0. We shall assume that φ and ψ belong to H_0.

The Bubnov-Galerkin method for nonstationary problems ([6]) consists in the following: one chooses a linearly independent complete coordinate system of elements $\{\varphi_k\}$ belonging to the space H_0. The approximate solution of problem (1), (2) is sought in the form

$$u_n(t) = \sum_{k=1}^{n} C_k^{(n)}(t)\varphi_k, \tag{3}$$

where the coefficients $C_k^{(n)}(t)$ are determined from the system of ordinary differential equations

$$\sum_{k=1}^{n} \frac{d^2 C_k^{(n)}(t)}{dt^2}[\varphi_k,\varphi_j]_A + \sum_{k=1}^{n} \frac{dC_k^{(n)}(t)}{dt}[\varphi_k,\varphi_j]_B + \sum_{n=1}^{n} C_k^{(n)}(t)[\varphi_k,\varphi_j]_C = (f,\varphi_j)$$

$$(j = 1, 2, \dots, n) \tag{4}$$

under the initial conditions

$$C_k^{(n)}(t)\Big|_{t=0} = C_k^{(n)}, \qquad \left.\frac{dC_k^{(n)}(t)}{dt}\right|_{t=0} = \dot{C}_k^{(n)}. \tag{4'}$$

The values $C_k^{(n)}$ and $\dot{C}_k^{(n)}$ may be prescribed to a considerable extent arbitrarily (see (6)), but we shall determine them in the following way: let $\varphi^{(n)}$ and $\psi^{(n)}$ denote the projections of the elements φ and ψ in the space H_0 onto the subspace of the elements $\varphi_1, \varphi_2, \dots, \varphi_n$. It is obvious that $\varphi^{(n)}$ and $\psi^{(n)}$ are linear combinations of the elements $\varphi_1, \varphi_2, \dots, \varphi_n$; we shall take the coefficients of these combinations as $C_k^{(n)}$ and $\dot{C}_k^{(n)}$. Obviously, these coefficients are determined from the conditions

$$\|\varphi - \varphi^{(n)}\|_0^2 = \left\| \varphi - \sum_{k=1}^{n} C_k^{(n)} \varphi_k \right\|_0^2 = \min,$$

$$\|\psi - \psi^{(n)}\|_0^2 = \left\| \psi - \sum_{k=1}^{n} \dot{C}_k^{(n)} \varphi_k \right\|_0^2 = \min .$$

Finding the minima of these functionals leads to the solution of systems of algebraic equations

$$\sum_{k=1}^{n} [\varphi_k, \varphi_j]_0 \, C_k^{(n)} = [\varphi, \varphi_j]_0,$$

$$\sum_{k=1}^{n} [\varphi_k, \varphi_j]_0 \, \dot{C}_k^{(n)} = [\varphi, \psi_j]_0 \qquad (j = 1, 2, \dots, n).$$

Obviously, these systems are solvable.

Suppose that the coefficients $[\varphi_k, \varphi_j]_A$, $[\varphi_k, \varphi_j]_B$, $[\varphi_k, \varphi_j]_C$ and the free terms (f, φ_j) are computed with small errors, respectively $a_{kj} = \bar{a}_{jk}$, $\beta_{kj} = \bar{\beta}_{jk}$, $\gamma_{kj} = \bar{\gamma}_{jk}$, and $\Delta_j^{(n)}(t)$. In addition, we shall assume that the initial values $C_j^{(n)}$ and $\dot{C}_j^{(n)}$ are also computed with small errors δ_j^0 and δ_j, respectively. Then, instead of the Bubnov-Galerkin system (4), we obtain the system

$$\sum_{k=1}^{n} \frac{d^2 \widetilde{C}_k^{(n)}(t)}{dt^2} \{[\varphi_k, \varphi_j]_A + a_{kj}\} + \sum_{k=1}^{n} \frac{d \widetilde{C}_k^{(n)}(t)}{dt} \{[\varphi_k, \varphi_j]_B + \beta_{kj}\} +$$

$$+ \sum_{k=1}^{n} \widetilde{C}_k^{(n)}(t) \{[\varphi_k, \varphi_j]_C + \gamma_{kj}\} = (f, \varphi_j) \qquad (j = 1, 2, \dots, n), \tag{5}$$

where $\widetilde{C}_k^{(n)}(t)$ is the solution with error.

Denote by $\Gamma_{n0}, \Gamma_{n1}, \Gamma_{n2}$ the matrices with elements $a_{kj} = \bar{a}_{jk}$, $\beta_{kj} = \bar{\beta}_{jk}$, $\gamma_{kj} = \bar{\gamma}_{jk}$, and by $\Delta_n(t)$, δ_{n0}, δ_{n1}, $C_n(t)$, and $\widetilde{C}_n(t)$ the n-dimensional vectors with components $(\Delta_1^{(n)}(t), \dots, \Delta_n^{(n)}(t))$, $(\delta_1^0, \dots, \delta_n^0)$, $(\delta_1, \dots, \delta_n)$, $(C_1^{(n)}(t), \dots, C_n^{(n)}(t))$, and $(\widetilde{C}_1^{(n)}(t), \dots, \widetilde{C}_n^{(n)}(t))$.

Since the coefficients, the free terms of the Bubnov-Galerkin system, and also the initial values are computed approximately, the question of stability under small changes of the coefficients, free terms, and initial values naturally arises. In this connection we introduce a definition.

Definition. We shall call the Bubnov–Galerkin method **stable on the finite interval** $0 \le t \le l$ if there exist constants p_i $(i = 0, 1, 2, 3, 4, 5)$, independent of n, such that, for sufficiently small norms of the matrices $\|\Gamma_{n0}\|$, $\|\Gamma_{n1}\|$, $\|\Gamma_{n2}\|$ and the following norms of the vectors $\|\Delta_n(t)\|_{L_2[0,l]}$, $\|\delta_{n0}\|$, $\|\delta_{n1}\|$, the inequality

$$\|\widetilde{C}_n(t) - C_n(t)\| \le p_0\|\delta_{n0}\| + p_1\|\delta_{n1}\| + p_2\|\Gamma_{n0}\| + + p_3\|\Gamma_{n1}\| + p_4\|\Gamma_{n2}\| + p_5\|\Delta_n(t)\|_{L_2[0,l]}, \tag{6}$$

holds for $0 \le t \le l$, where the norm $\|\cdot\|$ of a vector is defined by

$$\|C_n(t)\| = \left(\sum_{k=1}^{n} |C_k^{(n)}(t)|^2\right)^{1/2}.$$

Stability on the infinite interval $0 \le t < +\infty$ is defined in an analogous way.

Theorem 1. Suppose $A = 0$, $B = I$, and the coordinate system is normal[4] in the space H.

Then the Bubnov–Galerkin method is stable on the infinite interval of time.

Theorem 2. Suppose $A = I$, $B = aI$, where $a = \text{const} > 0$, and suppose the coordinate system $\{\varphi_k\}$ is normal in H.

Then the Bubnov–Galerkin method is stable on the infinite interval of time.

Theorem 3. Suppose $A = I$, $B = 0$, and the right-hand side $f(t)$ is such that

$$\int_0^\infty t^2 \|f(t)\|\, dt < +\infty.$$

Suppose, moreover, that the coordinate system $\{\varphi_k\}$ is normal in H.

Then the Bubnov–Galerkin method is stable on the infinite interval of time.

Theorem 4. Assume that the operators A, B, C are self-adjoint and positive definite in H, and that the spaces H_A, H_B, H_C consist of the same elements. Suppose the coordinate system $\{\varphi_k\}$ is normal in H_A.

Then the Bubnov–Galerkin method is stable on a finite interval of time.

Theorem 5. Suppose $A = 0$, and the operators B and C are self-adjoint and positive definite in H, and suppose $H_C \subset H_B$. If the coordinate system is normal in H_B, then the Bubnov–Galerkin method is stable on the infinite interval of time.

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Note: Figure translations are in progress. See original paper for figures.