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Abstract

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S. A. TERSENOV

ON A PROBLEM WITH DATA ON THE LINE OF DEGENERATION FOR SYSTEMS OF EQUATIONS OF HYPERBOLIC TYPE

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As is known (see (1)), the Cauchy problem with data on the line of degeneration of type for equations of hyperbolic type is, generally speaking, not well posed. In the present article this problem is studied in a modified formulation proposed in (1)*.

1. Consider the system of equations of hyperbolic type

$$\begin{aligned} u_x + v_y + au + bv &= f_1, \\ yu_y + v_x + cu + dv &= f_2, \end{aligned} \qquad (y > 0) \tag{1}$$

in the domain D, formed by the segment AB of the x-axis (the line of degeneration) and the characteristics:

$$x - 2\sqrt{y} = C, \qquad x + 2\sqrt{y} = C,$$

passing through the points A and B, respectively. Suppose that the coefficients and the right-hand sides of system (1) are defined and continuous in a certain rectangle of height δ, entirely containing the domain $\overline{D}$.

Introduce the function

$$\eta(x, y) = \exp \int_y^\delta c(x, \tau)\tau^{-1} d\tau.$$

First consider the case when $\eta(x, y)$ is integrable with respect to y in the interval $0 \leq y \leq \delta$ for any x from AB. It is easy to see that in this case $c(x, 0) \leq 1$. Denote by p the integer part of $\max(1 - c(x, 0))$ on AB. In addition suppose that the coefficients and the right-hand sides of system (1) have, in $\overline{D}$, continuous derivatives with respect to x up to order $(2p + 1)$, and that the initial data $\tau(x)$, $\nu(x)$ have continuous derivatives of order $(2p + 2)$.

Under the assumptions adopted, the following is valid.

Theorem 1. *There exists a unique continuously differentiable solution of system (1) in the domain D, satisfying the condition*

$$\eta^{-1}(u-\Phi)=\tau(x), \qquad v'_x(x,0)=\nu(x) \quad \text{on } AB, \tag{2}$$

where Φ is a completely determined function, which is constructed explicitly below.

By introducing new unknown functions

$$\begin{aligned} u_1 &= \sqrt{y}\,\eta\tau+\sqrt{y}\,\Phi+v+\Psi+\tfrac{1}{2}\sqrt{y}\,u+v,\\ u_2 &= \sqrt{y}\,\eta\tau+\sqrt{y}\,\Phi-v-\Psi+\sqrt{y}\,u-v \end{aligned}$$

the problem under consideration is obviously reduced to a system of integral equations (see (2)):

$$u_1=\int_0^{s_1}\left[t^{-1/2}\varphi+\frac{1-2c}{4}t^{-1}(u_1+u_2)+Au_1+Bu_2\right]dt,$$

$$u_2=\int_0^{s_2}\left[t^{-1/2}\varphi+\frac{1-2c}{4}t^{-1}(u_1+u_2)+A_1u_1+B_1u_2\right]dt, \tag{3}$$

where s_1, s_2 are the lengths of the arcs, respectively, of the characteristics $x-2\sqrt{y}=C,\ x+2\sqrt{y}=C$, passing through the point (x,y);

$$A=-\frac{1}{2}y^{-1/2}(a+d)-\frac{1}{2}b; \qquad B=-\frac{1}{2}y^{-1/2}(a-d)+\frac{1}{2}b;$$

$$A_1=\frac{1}{2}y^{-1/2}(a-d)+\frac{1}{2}b; \qquad B_1=\frac{1}{2}y^{-1/2}(a+d)-\frac{1}{2}b;$$

$$\varphi_i=-\eta\int_y^{\delta}t^{-1}\eta^{-1}\mu_i\,dt, \quad i=1,2,\dots,p-1; \qquad \varphi_p=\eta\int_0^{y}t^{-1}\eta^{-1}\mu_p\,dt;$$

$$\psi_i=\exp\left(-\int_0^{y}b\,dt\right)\int_0^{y}\exp\left(\int_0^{t}b\,d\tau\right)\gamma_i\,dt, \quad i=1,2,\dots,p;$$

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[*] This work was carried out in 1962 and reported at the seminar of the Department of Function Theory of the Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR.

$$\mu_1 = f_2 - v' - dv; \qquad \mu_i = -\psi_{1x} - d\psi_1, \quad i = 2, \dots, p;$$

$$\gamma_1 = f_1 - \eta\tau' - \eta_x\tau - a\eta\tau - v' - bv - \varphi_{1x} - a\varphi_1;$$

$$\gamma_i = -\varphi_{ix} - a\varphi_i, \quad i = 2, \dots, p;$$

$$\Phi = \sum_{i=1}^{p} \varphi_i, \qquad \Psi = \sum_{i=1}^{p} \psi_i, \qquad \varphi = -\psi_{px} - d\psi_p.$$

The functions φ_i, ψ_i satisfy the equations:

$$y\varphi_{iy} + c\varphi_i = \mu_i, \qquad \psi_{iy} + b\psi_i = \gamma_i, \quad i = 1, 2, \dots, p.$$

For small values of y, system (3) has a unique solution. The existence of a solution follows from the fact that $\varphi = O(1)y^{2-c(x,0)-\varepsilon}$ (see ([2])), where $\varepsilon > 0$ is a sufficiently small number. The uniqueness of the solution of system (3) follows from the fact that the solutions of the homogeneous problem ($f_i = \tau = v \equiv 0$) admit the estimates

$$u = o(1)y^{2-c(x,0)-\varepsilon}, \qquad v = o(1)y^{2-c(x,0)-\varepsilon};$$

O and o are Landau symbols.

Let now the function $\eta(x, y)$, for all values of x from AB, be nonintegrable in the interval $0 \le y \le \delta$. In this case $c(x, 0) \ge 1$. Denote by q the integer part of $\max c(x, 0)$ on AB. Suppose that the coefficients and right-hand sides of system (1), and the initial data $\tau(x)$ and $v(x)$, have continuous derivatives with respect to x up to order $(2q + 2)$ in $\overline{D}$. Then the following holds.

Theorem 2. a) There exists a unique continuously differentiable solution of system (1) in the domain D, satisfying the condition

$$\lim_{y\to 0} \frac{yu}{\omega} = v(x), \qquad \lim_{y\to 0}(v - \omega\Phi) = \tau(x) \quad \text{on } AB;$$

β) there exists a unique bounded solution of system (1), satisfying the condition

$$v(x, 0) = \tau(x) \quad \text{on } AB.$$

The proof of this theorem, by the substitution $u = y^{-1}\omega u_0,\ v = \omega v_0$, reduces to the preceding theorem.

The function $\Phi(x, y)$ is determined in the manner indicated above after the introduction of the new unknown functions u_0, v_0;

$$\omega(x, y) = \int_y^\delta \eta(x, t)\, dt + 1.$$

Remark. Theorems 1 and 2 also hold for the system

$$u_x + v_y + au + bv = f_1,$$

$$y^n u_y + v_x + y^{mcu} + dv = f_2,$$

where $1 < n < 2,\ m \geq n - 1$.

2. We now consider the system of equations of hyperbolic type

$$y^n u_x + v_y + au + bv = f_1,$$

$$u_y + v_x + cu + dv = f_2, \qquad (y > 0,\ n > 0) \tag{4}$$

in the domain D, bounded by the segment AB of the x-axis (the line of degeneracy) and by the characteristics:

$$x - \frac{2}{2+n} y^{1+n/2} = C, \qquad x + \frac{2}{2+n} y^{1+n/2} = C,$$

passing through the points A and B, respectively.

If the function $2a_0(x, y) = y^{1-n/2} a(x, y)$ is unbounded in a neighborhood of the segment, then the Cauchy problem, generally speaking, is not well posed (see ($^1, ^3$)). If $a_0(x, y) = o(1)$, then the solution of the Cauchy problem (see ($^1, ^4$)) always exists and is unique. In paper (^{1}) it was proposed to study the Cauchy problem under the condition $a_0(x, y) = O(1)$.

Let $a_0(x, y)$ be a continuous function in $\overline{D}$, and let

$$\sigma = \frac{\max |a_0(x, 0)| + 2}{4} \quad \text{on } AB.$$

Denote by ρ the integer part of σ, if σ is not an integer, and $\rho = \sigma + 1$, if σ is an integer.

Suppose that the coefficients of system (4) have continuous derivatives with respect to x up to order 2ρ in $\overline{D}$, and that the initial data $\tau(x)$ and $\nu(x)$ have continuous derivatives of order $2\rho+1$ on AB. Under these assumptions the following holds.

Theorem 3. *There exists a unique continuously differentiable solution in the domain D of system (4), satisfying the condition*

$$u(x,0)=\tau(x), \qquad v(x,0)=\nu(x).$$

By introducing the new unknown functions

$$u_1=y^{n/2}\tau+y^{n/2}\Phi+\nu+\Psi+y^{n/2}u+v;$$

$$u_2=y^{n/2}\tau+y^{n/2}\Phi-\nu-\Psi+y^{n/2}u-v,$$

as in the preceding cases, the problem is reduced to the system of integral equations:

$$u_1=\int_0^{s_1}\left[t^{n/2}\varphi+\frac{n-a_0}{4}t^{-1}(u_1+u_2)+Au_1+Bu_2\right]dt,$$

$$(5)$$

$$u_2=\int_0^{s_2}\left[t^{n/2}\varphi+\frac{n+a_0}{4}t^{-1}(u_1+u_2)+A_1u_1+B_1u_2\right]dt,$$

where

$$\begin{gathered}A=-\frac{1}{2}(c+b+dy^{n/2}); \qquad B=-\frac{1}{2}(c-b-dy^{n/2});\\ A_1=-\frac{1}{2}(c-b+dy^{n/2}); \qquad B_1=-\frac{1}{2}(c+b-dy^{n/2});\\ \Phi=\sum_{s=1}^{\rho}\varphi_s; \qquad \Psi=\sum_{s=1}^{\rho}\psi_s; \qquad \varphi=-\psi_{\rho x}-d\psi_\rho;\end{gathered}$$

φ_s,ψ_s are solutions of the recurrent system of equations:

$$\varphi_{sy}+c\varphi_s=\mu_s, \qquad \psi_{sy}+b\psi_s=\gamma_s$$

under the condition

$$\varphi_s(x,0) = \psi_s(x,0) = 0; \tag{6}$$

$$\mu_1 = f_2 - c\tau - \nu' - d\nu; \qquad \mu_s = -\psi_{s-1,x} - d\psi_{s-1}, \quad s = 2, \dots, \rho;$$
$$\gamma_1 = f_1 - y^n\tau' - a\tau - b\nu - y^n\varphi_{1x} - a\varphi_1;$$
$$\gamma_s = -y^n\varphi_{sx} - a\varphi_s, \quad s = 2, \dots, \rho.$$

The existence and uniqueness of a solution of system (5) follows from the fact that $\varphi = O(y^{2\rho})$ and that the solution of the homogeneous problem ($f_i = \tau = \nu = 0$) admits the estimate:

$$u = o(1)y^{2\rho+n/2+1}, \qquad v = o(1)y^{2\rho+n/2+1}.$$

The latter conditions can be obtained from the homogeneous system corresponding to (4).

3. A statement analogous to Theorem 3 also holds for a system of equations of hyperbolic type of second order:

$$y^n u_{xx} - u_{yy} + au_x + bu_y + cu = f \qquad (y > 0,\ n > 0), \tag{7}$$

if the matrix $y^{1-n/2}a$ is continuous in the characteristic triangle D containing the segment AB of the x-axis (the line of degeneracy).

Denote by a_{ij} the elements of the matrix a. Let

$$\sigma = \frac{1}{2} + \frac{1}{4}\max_{i,j,x}\{y^{1-n/2}a_{ij}\}_{y=0},$$

and let ρ be the integer part of σ, if σ is not an integer, and $\rho = \sigma + 1$, if σ is an integer.

Theorem 4. *If the matrices a, b, c, and the vector f are continuous and have continuous derivatives with respect to x up to order $(2\rho+1) - 2\rho$ in $\overline{D}$, and the initial data $\tau(x)$ and $\nu(x)$ on AB are continuously differentiable functions up to order $2\rho+3$, then there exists a unique twice continuously differentiable solution in the domain D of system (7), satisfying the condition*

$$u(x,0) = \tau(x), \qquad v(x,0) = \nu(x) \quad \text{on } AB.$$

The proof of this theorem is entirely analogous to the proof of Theorem 2 (see ([2],[4])).

Institute of Mathematics with Computing Center
Siberian Branch of the Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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