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Abstract

Full Text

Mathematics

A. S. Markus

On Some Criteria for Completeness of the System of Root Vectors of a Linear Operator and Summability of Series with Respect to This System

(Presented by Academician M. V. Keldysh, December 10, 1963)

The main purpose of the present note is to establish some criteria for completeness of the system of root vectors of a linear operator in a Banach space and for summability of series with respect to this system by Abel's method in the sense of V. B. Lidskii ⁽¹⁾. These results are presented in §§ 1-3. In § 4 a Hilbert space is considered; there some supplements are given to the theorems of M. V. Keldysh ⁽²⁾, V. B. Lidskii ^(3,4), and V. I. Matsaev ⁽⁵⁾ on completeness and to the theorem of V. B. Lidskii ⁽¹⁾ on summability. In § 5 one application of the results of § 1 to elliptic operators in the spaces L_p ($p > 1$) is indicated.

1. Let \mathfrak{B} be a separable Banach space, \mathfrak{A} the normed ring of all linear bounded operators acting in \mathfrak{B} , and \mathfrak{S}_∞ the two-sided ideal of completely continuous operators. If $A \in \mathfrak{S}_\infty$, then by $\{\lambda_j(A)\}$ we denote the complete system of eigenvalues of the operator A , numbered in nondecreasing order of modulus and with account of their multiplicities. For $A \in \mathfrak{S}_\infty$ set

$$s_n(A) = \inf_{K \in \mathfrak{A}_{n-1}} \|A - K\| \quad (n = 1, 2, \dots),$$

where \mathfrak{A}_n is the set of all n -dimensional operators from \mathfrak{A} . As I. Ts. Gohberg and M. G. Krein showed, in the case of a Hilbert space this definition is equivalent to the usual one:

$$s_n(A) = \lambda_n((A^*A)^{1/2}) \quad (n = 1, 2, \dots) \quad (6).$$

By \mathfrak{S}_p ($0 < p < \infty$) (following the notation adopted in ⁽⁶⁾ for a Hilbert space) we shall denote the set of all operators $A \in \mathfrak{S}_\infty$ for which

$$\sum_j s_j^p(A) < \infty,$$

and by $\mathfrak{S}_\infty^{(0)}$ the set of all operators $A \in \mathfrak{S}_\infty$ for which $s_j(A) \rightarrow 0$, i.e. the closure of the set of finite-dimensional operators in the ring \mathfrak{R} . By \mathfrak{N} (respectively \mathfrak{N}_β ($\beta > 1$)) we denote the set of all operators $A \in \mathfrak{R}$ representable in the form

$$Ax = \sum_{j=1}^{\infty} \varepsilon_j f_j(x) y_j \quad (\|f_j\| = \|y_j\| = 1; j = 1, 2, \dots),$$

where

$$\sum |\varepsilon_j| < \infty$$

(respectively $\varepsilon_j = o(j^{-\beta})$). Operators $A \in \mathfrak{N}$ are called nuclear.

It is easy to see that \mathfrak{S}_p ($p > 0$), $\mathfrak{S}_\infty^{(0)}$, \mathfrak{N} , and \mathfrak{N}_β ($\beta > 1$) are two-sided ideals of the ring \mathfrak{R} .

By $\mathfrak{E}(A)$ we denote the linear closed span of all root vectors of the operator $A \in \mathfrak{S}_\infty$, and by $\tilde{\mathfrak{E}}(A)$ the linear closed span of its root vectors corresponding to eigenvalues different from zero. Following ⁽²⁾, we shall call an operator A complete if $\tilde{\mathfrak{E}}(A) = \mathfrak{B}$.

By $\mathfrak{R}(A)$ and $\mathfrak{Z}(A)$ we further denote the range and the kernel of the operator A :

$$\mathfrak{R}(A) = A\mathfrak{B}, \quad \mathfrak{Z}(A) = A^{-1}(0).$$

Theorem 1. Let the operator $H \in \mathfrak{R}$ have a purely real spectrum and let its resolvent for nonreal λ admit the estimate

$$\|(H - \lambda I)^{-1}\| \leq C |\operatorname{Im} \lambda|^{-1}. \quad (1)$$

If, for some natural number n , the operator H^n is nuclear, and the operator $T \in \mathfrak{S}_\infty^{(0)}$, $B = (I + T)H$, and $\mathfrak{R}(B) = \mathfrak{B}$, then the operator B is complete.

This theorem is a generalization of a theorem of M. V. Keldysh ⁽²⁾ on the completeness of the operator acting in Hilbert space

$$B = H(I + T),$$

where $H = H^* \in \mathfrak{S}_p$, $T \in \mathfrak{S}_\infty$, and $\mathfrak{Z}(B) = 0$.

We note that the condition $H^n \in \mathfrak{R}$ in Theorem 1 may be replaced by either of the following conditions:

- 1) $H \in \mathfrak{S}_p$ ($0 < p < \infty$);

$$2) \sum_j |\lambda_j(H)|^p < \infty \quad (0 < p < \infty).$$

Theorem 2. Let the operator A satisfy the conditions:

- a) $A \in \mathfrak{M}_\beta$ ($1 < \beta < 5/2$);
- b) all points λ lying outside the angle

$$\Lambda = \{z : -\gamma\pi/2 \leq \arg z \leq \gamma\pi/2\}$$

($\gamma = \beta - 1/2$) are regular points of the operator A , and

$$\|(A - \lambda I)^{-1}\| C[\rho(\lambda, \Lambda)]^{-1} \quad (\lambda \in \bar{\Lambda}), \quad (2)$$

where $\rho(\lambda, \Lambda)$ is the distance from the point λ to Λ .

Then $\mathfrak{C}(A) = \mathfrak{R}(A)$, and if $\mathfrak{B} = \mathfrak{B}^{**}$, then $\mathfrak{C}(A) = \mathfrak{B}$.

Theorem 3. Let the operator A satisfy condition a) of Theorem 2 and condition b) of the same theorem, but with $\gamma < \beta - 1/2$. If the operator $T \in \mathfrak{S}_\infty^{(0)}$, $B = (I + T)A$, and $\mathfrak{R}(B) = \mathfrak{B}$, then the operator B is complete; moreover, for every vector $f \in \mathfrak{R}(B)$ the corresponding series in the system of root vectors of the operator B is summable to f by the method (A, λ, α) (in the sense of V. B. Lidskii ⁽¹⁾) for

$$\gamma^{-1} > \alpha \geq (\beta - 1/2)^{-1}.$$

We note that Theorem 2 remains valid if condition a) is replaced by the condition $A^n \in \mathfrak{R}$ and one puts $\gamma = 1/2n$, and Theorem 3—if condition a) is replaced by $A \in \mathfrak{R}$ and one assumes $\gamma < 1/2$.

The proofs of Theorems 1–3 are based on the following theorem, which is established with the aid of important results of Gohberg ^(7,8) on nuclear operators.

Theorem 4*. For every nuclear operator A ,

$$\lim_{\rho \rightarrow \infty} \left\{ \rho^{-2} \ln \max_{|\lambda|=\rho} \|(I - \lambda A)^{-1}\| \right\} = 0.$$

If, however, $A \in \mathfrak{M}_\beta$ ($\beta > 1$) and $r = (\beta - 1/2)^{-1}$, then

$$\lim_{\rho \rightarrow \infty} \left\{ \rho^{-r} \ln \max_{|\lambda|=\rho} \|(I - \lambda A)^{-1}\| \right\} = 0. \quad (3)$$

An essential role in the proofs of Theorems 1, 3 and of Theorem 8 below is played by the following

Lemma. Let $A \in \mathfrak{S}_\infty$ be a complete operator, M an unbounded set in the complex plane, and

$$\|(I - \lambda A)^{-1}\| \leq C_0 \quad (\lambda \in M, |\lambda| \geq \rho_0).$$

If $T \in \mathfrak{S}_\infty^{(0)}$ and $\mathfrak{Z}(I + T) = 0$, then

$$\|(I - \lambda(I + T)A)^{-1}\| \leq C_1 \quad (\lambda \in M, |\lambda| \geq \rho_1).$$

2. Here we shall indicate a criterion for the validity of inequalities of the form (1)-(2) with $C = 1$.

Define a mapping G of the space \mathfrak{B} onto \mathfrak{B}^* as follows: by G_x ($x \in \mathfrak{B}$) denote some functional from \mathfrak{B}^* possessing the pro-

* In discussing the results of this paper with V. I. Matsaev it became clear that he had obtained certain resolvent estimates for operators acting in Banach space; from these estimates, in particular, it follows that for $A \in \mathfrak{S}_p$ ($0 < p < \infty$) and any $r > p$, (3) holds.

properties: $G_x(x) = \|x\|^2$ and $\|G_x\| = \|x\|^*$. The **numerical range** of an operator $A \in \mathfrak{A}$ is called, following Lumer ⁽⁹⁾, the set $W(A)$ of all complex numbers of the form $G_x(Ax)$, where $x \in \mathfrak{B}$, $\|x\| = 1$. It is easy to see that in the case of a Hilbert space \mathfrak{H} this definition coincides with the usual definition of $W(A)$ as the set of all numbers of the form (Af, f) ($f \in \mathfrak{H}$, $\|f\| = 1$) (see ⁽¹⁰⁾).

As was established in ⁽⁹⁾, the convex hull of the set $W(A)$ does not depend on the choice of the mapping G .

Theorem 5. *Let $A \in \mathfrak{A}$, and let F be some closed convex set in the complex plane. In order that all points $\lambda \notin F$ be regular for A and that the estimate*

$$\|(A - \lambda I)^{-1}\| \leq [\rho(\lambda, F)]^{-1} \quad (\lambda \notin F),$$

hold, it is necessary and sufficient that $F \supset W(A)$.

Remark. *For an operator $A \in \mathfrak{A}$ the following assertions are equivalent:*

- 1°. $\|(A - \lambda I)^{-1}\| \leq |\operatorname{Im} \lambda|^{-1}$ ($\operatorname{Im} \lambda \neq 0$).
- 2°. $W(A)$ lies on the real axis.
- 3°. $\|I + i\alpha A\| = 1 + o(\alpha)$ ($\alpha \rightarrow 0$, α real).

The equivalence of assertions 1° and 2° follows from Theorem 5, and the equivalence of 2° and 3° was established in ⁽⁹⁾. We note that property 3° was taken by Vidav ⁽¹¹⁾ as the definition of a Hermitian operator in a Banach space.

3. Recall that an operator $U \in \mathfrak{A}$ is called a **contraction** if $\|U\| \leq 1$.

Theorem 6. Let $A \in \mathfrak{M}_{3/2}$ and let $I + A$ be a contraction. Then $\mathfrak{C}(A) = \overline{\mathfrak{R}(A)}$, and if $\mathfrak{B} = \mathfrak{B}^{**}$, then $\mathfrak{C}(A) = \mathfrak{B}$.

Theorem 7. Let $A \in \mathfrak{M}_\beta$ ($\beta > 3/2$) and $I + A$ be a contraction. If $T \in \mathfrak{S}_\infty^0$, $B = (I + T)A$, and $\mathfrak{R}(B) = \mathfrak{B}$, then the operator B is complete and, moreover, for every vector $f \in \mathfrak{R}(B)$ the corresponding series in the system of root vectors of the operator B is summable to f by the method (A, λ, α) for $1 > \alpha \geq (\beta - 1/2)^{-1}$.

Theorems 6 and 7 follow from Theorems 2, 3, and 5.

4. In this section it is assumed that $\mathfrak{B} = \mathfrak{H}$ is a separable Hilbert space.

Theorem 8. Let the operator $A \in \mathfrak{S}_\infty$ satisfy the following conditions:

- a) $s_n(A) = o(n^{-\beta})$ ($\beta > 0$);
- b) $W(A)$ is contained in the angle

$$\Lambda = \{z : -\gamma\pi/2 \leq \arg z \leq \gamma\pi/2\},$$

where $\gamma < \min(\beta, 2)$.

If $T \in \mathfrak{S}_\infty$, $B = A(I + T)$, and $\mathfrak{Z}(B) = 0$, then the operator B is complete and, moreover, for every vector $f \in \mathfrak{R}(B)$ the corresponding series in the system of root vectors of the operator B is summable to f by the method (A, λ, α) for $\gamma^{-1} > \alpha \geq \beta^{-1}$.

This theorem is a generalization of Theorem 3 of V. B. Lidskii ⁽¹⁾, which is obtained from the formulated theorem for $T = 0$.

Theorem 8 remains valid if condition b) is replaced by the following:

b') all points λ lying outside the angle Λ are regular for A , and

$$\|(A - \lambda I)^{-1}\| \leq C[\rho(\lambda, \Lambda)]^{-1} \quad (\lambda \notin \Lambda).$$

As for the conclusion on the completeness of the operator B , it remains valid also if condition b) is replaced by the condition

b'') the points λ lying outside all the angles

$$\Lambda_j = \{z : \varphi_j \leq \arg z \leq \psi_j\} \quad (j = 1, 2, \dots, n),$$

where

$$0 \leq \varphi_1 \leq \psi_1 < \varphi_2 \leq \psi_2 < \dots < \varphi_n \leq \psi_n < 2\pi; \quad \psi_j - \varphi_j \leq \gamma\pi \quad (j = 1, 2, \dots, n),$$

are regular points of the operator A , and

$$\|(A - \lambda I)^{-1}\| \leq C \left[\min_{1 \leq j \leq n} \rho(\lambda, \Lambda_j) \right]^{-1} \quad \left(\lambda \notin \bigcup_1^n \Lambda_j \right).$$

* The functional G_x is uniquely determined by x if and only if the norm in \mathfrak{B} is Gateaux differentiable; under this condition $G_x = \|x\|\Gamma x$, where Γx is the gradient of the norm.

The completeness theorem with conditions a) and b'') generalizes M. V. Keldysh' s theorem ⁽²⁾ on the completeness of the operator $H(I+T)$, since for the operator $H = H^*$ condition b'') is fulfilled for $n = 2$, $\varphi_1 = -\psi_1 = 0$ and $\varphi_2 = -\psi_2 = \pi$.

Theorem 9. *If $H = H^* \in \mathfrak{S}_\infty$, $T \in \mathfrak{S}_\infty$, $\sum_n n^{-1}|\lambda(T-T^*)| < \infty$, $B = H(I+T)$, and $\mathfrak{Z}(B) = 0$, then the operator B is complete.*

Theorem 9 is a certain strengthening of Theorem 3 of V. I. Macaev ⁽⁵⁾ and is proved without difficulty on the basis of that theorem.

For lack of space we do not give the completeness and summability propositions for unbounded operators and linear operator pencils that follow from Theorems 1-3 and 6-9 or are proved analogously to them.

5. Let D be a bounded domain with sufficiently smooth boundary S in n -dimensional Euclidean space, and let $L_p^{(N)}(D)$ be the space of N -dimensional vector-functions $u = \{u_j\}_1^N$ ($u_j \in L_p(D)$; $\|u\|^p = \sum_j \|u_j\|^p$).

Theorem 10. *Let \mathcal{L} be a strongly elliptic differential expression of order $2m$ with sufficiently smooth coefficients. If $m \geq n$, where n is the number of independent variables, and $p \geq 1$, then the problem*

$$\begin{aligned} \frac{du}{dt} + \mathcal{L}u &= 0, & u|_{t=0} &= f \ (\in L_p^{(N)}(D)), \\ u|_S &= \frac{\partial u}{\partial \nu} \Big|_S = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \Big|_S = 0 \end{aligned} \quad (4)$$

has a solution that expands into an absolutely convergent series in $L_p^{(N)}(D)$

$$u(t) = \sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} c_j(t) g_j \right) \quad (t > 0)$$

with respect to the system $\{g_j\}_1^\infty$ of root vectors of the operator \mathcal{L}_0 , generated in the space $L_p^{(N)}(D)$ by the differential expression \mathcal{L} and the boundary conditions (4).

Theorem 10 is proved with the aid of the theorems of § 1 and certain results of A. Grothendieck ⁽⁷⁾, M. Z. Solomyak ⁽¹²⁾, and D. M. Eidus ⁽¹³⁾. For $p = 2$ this theorem was established by V. B. Lidskii ⁽¹⁾ under the more general condition $2m > n$.

From the results of § 1 one can draw the conclusion about the completeness in $L_p^{(N)}(D)$ ($p > 1$) of the system of root vectors of the operator \mathcal{L}_0 and about the summability of series with respect to this system not only under the condition $m \geq n$, but also under certain other conditions, for example in the case of a single equation ($N = 1$). We note that in this case the completeness of the system of root vectors of an elliptic operator in $L_p(D)$ ($p > 1$) for a very broad class of boundary conditions, including conditions (4), was recently established by Agmon (¹⁴).

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Institute of Physics and Mathematics
Academy of Sciences of the Moldavian SSR

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