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Abstract

Full Text

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PROOF OF THE EXISTENCE OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES NOT REPRESENTABLE BY LINEAR SUPERPOSITIONS OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS OF A SMALLER NUMBER OF VARIABLES

(Presented by Academician A. N. Kolmogorov on 30 III 1964)

Let $G(\rho, z_1, z_2)$ be a domain in the space of two complex variables $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, defined by the inequalities $|y_i| \leq \rho$ $(i = 1, 2)$.

Theorem. For any functions $p_m = p_m(x_1, x_2)$, continuous in the whole plane, and functions $q_m = q_m(x_1, x_2)$ $(m = 1, 2, \ldots, N)$, continuously differentiable in the whole plane, and for any domain D of the plane of the variables x_1, x_2, there exist a number $\rho > 0$ and a function $f(z_1, z_2)$, analytic and bounded in the domain $G(\rho, z_1, z_2)$, taking real values on the plane of the variables x_1, x_2 $(y_1 = y_2 = 0)$, and not equal in the domain D to any superposition of the form

$$\sum_{m=1}^{N} p_m(x_1, x_2) f_m(q_m(x_1, x_2)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions.

It is already of interest to compare Theorem 1 with A. N. Kolmogorov's theorem on the possibility of representing every continuous function of two variables by a superposition of the form

$$\sum_{i=1}^{5} f_i(\alpha_i(x) + \beta_i(y)),$$

where all the functions are continuous, and $\{\alpha_i(x) + \beta_i(y)\}$ are fixed in advance.

Notation: $\omega(\delta)$ is the modulus of continuity of the functions $\{p_m; \partial q_m/\partial x_1; \partial q_m/\partial x_2\}$; $\mathrm{grad}[f(z)]$ is the gradient of $f(z)$; $d_1(e)$ is the one-dimensional diameter of the set e; $h_1(e)$ is the length of the set e; $c_1, c_2, \ldots$ are constants.

Lemma 1. In every domain D one can fix a closed subset G, which is the union of a finite number of simply connected closed domains, specify a constant

$c > 0$, and renumber the functions $\{p_m; q_m\}$ by two indices in such a way that the newly obtained functions

$$p_i^k = p_i^k(x_1, x_2), \qquad q_i^k = q_i^k(x_1, x_2)$$

$$(i = 0, 1, 2, \dots, n;\ k = 1, 2, \dots, m_i;\ \sum_{i=0}^{n} m_i \leq N),$$

i.e. some of the pairs of functions $\{p_m; q_m\}$ under this renumbering may in general be omitted, satisfy the following eight conditions:

1. $\{p_i^k; \partial q_i^k/\partial x_1; \partial q_i^k/\partial x_2\}$ have modulus of continuity $\omega(\delta)$.
2. $c \leq |p_i^k(x_1, x_2)| \leq c^{-1}$, $(x_1, x_2) \in G$.
3. For $i = 0$, $q_i^k \equiv \text{const}$ in G, and for $i > 0$, $c \leq |\operatorname{grad}[q_i^k(x_1, x_2)]| \leq c^{-1}$, $(x_1, x_2) \subset G$.
4. $q_i^k(x_1, x_2) \equiv \varphi_i^{kl}(q_i^l(x_1, x_2))$, $(x_1, x_2) \in G$, where $\varphi_i^{kl}(t)$ is a strictly monotone continuously differentiable function of t.
5. If $i \neq j$, then, for all k and l, the absolute value of the acute angle formed by the level lines of the functions q_i^k and q_j^l passing through an arbitrary point $(x_1, x_2) \in G$ does not exceed c.
6. The set G is the union of pairwise nonintersecting closed simply connected domains $\{G_l\}$, each of which is such that,

for all i, k the intersection of the level set of the function q_i^k with this domain is either the empty set or a simple arc (with endpoints on the boundary G_l) of length not less than c.

7. For every $i > 0$ and all functions $\{f_i^k(t)\}$ the inequality holds

$$\sup_{(x_1,x_2)\in G} \left| \sum_{k=1}^{m_i} p_i^k(x_1, x_2) f_i^k(q_i^k(x_1, x_2)) \right| \geq c \max_k \sup_{(x_1,x_2)\in G} \left| f_i^k(q_i^k(x_1, x_2)) \right|.$$

8. For any bounded measurable functions $\{\varphi_m(t)\}$ there exist measurable bounded functions $\{f_i^k(t)\}$ such that

$$\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k f_i^k(q_i^k) = \sum_{m=1}^{N} p_m \varphi_m(q_m) \quad \text{in } G.$$

We shall prove Lemma 1 by induction on N. For $N = 1$ the assertion of the lemma is easy to verify. Let $N > 1$. Fix a domain $G^* \subset D$ and renumber the functions $\{p_m; q_m\}$ by two indices so that the functions $\{p_i^k; q_i^k\}$ in the domain

G^* satisfy conditions 1)–5). Denote by $e_{i,t}$ the level set t of the function q_i^1; fix some set $\Gamma \subset G^*$ and put

$$\lambda_i(t,\Gamma) = \inf_{\{c_i^k\}} \sup_{(x_1,x_2)\in\Gamma\cap e_{i,t}} \left|\sum_{k=1}^{m_i} c_i^k p_i^k(x_1,x_2)\right|,$$

where the infimum is taken over all sets $\{c_i^k\}$ such that $\max_k |c_i^k| = 1$.

The domain of definition of the function $\lambda_i(t,\Gamma)$ is the set of values $t = q_i^1(x_1,x_2)$, $(x_1,x_2) \in \Gamma$. If Γ is closed, then $\lambda_i(t,\Gamma)$ is continuous. If for Γ $\lambda_i(t,\Gamma) = 0$, then there exists a subdomain $\Gamma^* \subset \Gamma$ and measurable functions $c_i^k(t)$, bounded by one, such that

$$\sum_{k=1}^{m_i} p_i^k(x_1,x_2)c_i^k(q_i^k(x_1,x_2)) \equiv 0$$

in Γ^*, and for some k

$$c_i^k(q_i^k(x_1,x_2)) \equiv 1$$

in Γ^*. But this means that one of the terms of the superposition

$$\sum_{k=1}^{m_i} p_i^k f_i^k(q_i^k)$$

can be excluded, without thereby narrowing the class of functions representable by superpositions of this kind (see condition 8). In this case Lemma 1 is proved by virtue of the corresponding induction hypothesis. Consequently, further we may assume that for every open set Γ the corresponding set $\lambda_i(t,\Gamma) > 0$ everywhere densely (and openly, by continuity of the function $\lambda_i(t,\Gamma)$) in $q_i^1(\Gamma)$. If for the set Γ

$$\lambda_i(t,\Gamma) \geq c_\Gamma = \text{const} > 0,$$

and Γ'' is such that for every $t \in q_i^1(\Gamma)$ the set $e_{i,t} \cap \Gamma''$ is an ε-net in $e_{i,t} \cap \Gamma$ (ε does not depend on t and is sufficiently small), then

$$\lambda_i(t,\Gamma\cap\Gamma'') \geq \tfrac{1}{2}c_\Gamma.$$

From the last two assertions it follows that one can indicate open sets, consisting of a finite number of components,

$$G^* \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \Gamma_{n+1},$$

such that, for every i,

$$\lambda_i(t,\Gamma_{i+1}) \geq c' = \text{const} > 0,$$

and each component of the set Γ_i has a boundary consisting of a finite number of segments of level lines of the functions $\{q_j^1\}$. Then, for every $i > 0$ and all functions $\{f_i^k(t)\}$,

$$\sup_{(x_1,x_2)\in\Gamma_n} \left|\sum_{k=1}^{m_i} p_i^k(x_1,x_2) f_i^k(q_i^k(x_1,x_2))\right| \geq c' \max_k \sup_{(x_1,x_2)\in\Gamma_n} \left|f_i^k(q_i^k(x_1,x_2))\right|,$$

i.e., condition 7 is fulfilled.

We take for G the closure of the set Γ_n. Fulfillment of condition 6 can be achieved already in defining the sets Γ_i, by requiring that for every i the boundary of every component γ of the set Γ_i consist of a finite number of segments of level lines of the functions $q_1^1, q_2^1, \dots, q_n^1$ such that, if the arc $[a,b]$ passes into the arc $[b,d]$ $(q_i^1([a,b]) =$

$=$ const and $q_m^1([b,d]) =$ const), then for every $k \neq j, m$ the level line of the function q_k^1 passing through the point b intersects the arc $[a,b,d]$, passing at the point b from the component γ to its complement. The lemma is proved.

Lemma 2. Let $[a', a'']$ and $[b', b'']$ be segments of level lines of the functions $\{q_i^k\}$ (i fixed); $\alpha = h_1([a', a''])$; $d_1(a' \cup b') \leq \delta$; $d_1(a'' \cup b'') \leq \delta$. Then, if $\{p_i^k\}$ and $\{q_i^k\}$ satisfy conditions 1-6 and δ is sufficiently small in comparison with α, then for all continuous functions $\{f_i^k(t)\}$

$$\left| \int_{s=[a',a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int_{s\in[b',b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds \right| \leq$$

$$\leq c_1(\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

$$\varepsilon = \max_{(x_1,x_2)\in G} \left| \sum_{i,k} p_i^k f_i^k(q_i^k) \right|; \qquad m = \max_{i,k} \max_{(x_1,x_2)\in G} |f_i^k(q_i^k)|,$$

c_1 does not depend on $\alpha, \delta, \varepsilon, m$.

Proof. On $[a', a'']$ fix a system of points $a_1, a_2, \dots, a_\nu$, uniformly distributed with respect to length ($a' = a_1$; $a'' = a_\nu$), and denote by b_r the point of intersection of the level line of the function q_i^k, containing the arc $[b', b'']$, with the level line of the function q_j^k passing through the point a_r (here $j \neq i$ should be regarded as fixed). By Lemma 3, from (1) we have

$$\left| \int_{s\in[a',a'']} p_j^k(s) f_j^k(q_j^k(s))\, ds - \int_{s\in[b',b'']} p_j^k(s) f_j^k(q_j^k(s))\, ds \right| =$$

$$= \lim_{\nu\to\infty} \left| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}]) - \right.$$

$$\left. - \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])(1 + O(1)\omega(\delta)) \right| +$$

$$+ O(1) m(\delta + \alpha\omega(\delta)) = O(1) m(\delta + \alpha\omega(\delta)).$$

Then

$$\left|\int_{s\in[a',a'']}\sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\,ds - \int_{s\in[b',b'']}\sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\,ds\right| \le$$

$$\le c_2\varepsilon\alpha + n\max_{j\neq i,\ k} m_j \left|\int_{s\in[a',a'']} p_j^k(s) f_j^k(q_j^k(s))\,ds - \int_{s\in[b',b'']} p_j^k(s) f_j^k(q_j^k(s))\,ds\right| \le$$

$$\le c_2\varepsilon\alpha + c_3 m(\delta + \alpha\omega(\delta)) \le c_1(\alpha\varepsilon + m\delta + m\alpha\omega(\delta)).$$

The lemma is proved.

Let $F = F(p, q, m, \varepsilon)$ be the set of superpositions of the form

$$f(x_1, x_2) = \sum_{i=0}^{n}\sum_{k=1}^{m_i} p_i^k(x_1, x_2)\, f_i^k(q_i^k(x_1, x_2))$$

such that

$$\max_{(x_1,x_2)\in G} |f(x_1, x_2)| \le \varepsilon',$$

$\{p_i^k\}$ and $\{q_i^k\}$ satisfy conditions 1-8, and $\{f_i^k\}$ are measurable and bounded by the constant m. Put

$$R(f(x_1, x_2), \delta) = \max_{S(\delta,x_1,x_2)} \left|\frac{1}{\pi\delta^2}\iint_{S(\delta,x_1,x_2)} f(u, v)\,du\,dv\right|,$$

where $S(\delta, x_1, x_2)$ is the disk of radius δ with center at the point (x_1, x_2). Denote by $\mathcal{H}^{\delta}_{\varepsilon^*}(F)$ the ε^*-entropy of the space F, taking as the distance between functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2) \in F$ the number

$$R(f_1(x_1, x_2) - f_2(x_1, x_2), \delta).$$

Lemma 3. If $0 < \theta \le 1$ and $m/\theta\varepsilon \ge 2$, then

$$\mathcal{H}^{\delta}_{\theta\varepsilon}(F(p, q, m, \varepsilon)) \le c_4(1/\theta^2\delta + \log m/\varepsilon),$$

where c_4 does not depend on $m, \varepsilon, \theta, \delta$; $\theta > A\omega(\delta)$.

Proof. Let $e_{i,j}$ be level sets of the functions q_i' $(i = 1, \dots, n;\ j = 1, 2, \dots, r_i)$ such that, for every i,

$$e_i = \bigcup_{j=1}^{r_i} e_{ij}$$

is a β-net in G (β will be fixed below). We partition the set F into the smallest possible number of subsets $F_1, \dots, F_r$ such that, for every ν and for any functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ from F_ν, their difference

$$\tilde{f}(x_1, x_2) = f_1(x_1, x_2) - f_2(x_1, x_2) = \sum_{i,k} p_i^k f_i^k(q_i^k)$$

is such that, for all i, k, j,

$$\left|\tilde{f}_i^k(q_i^k(e_{i,j}))\right| \le c_5\varepsilon.$$

For fixed c_5 and β,

$$r \le \frac{c_6 m}{\varepsilon}.$$

We shall show that

$$\mu = \max_{i,k} \max_{(x_1,x_2)\in G} \left|\tilde{f}_i^k(q_i^k(x_1, x_2))\right| \le c_7\varepsilon.$$

Suppose, for definiteness, that

$$\tilde{f}_1^1(q_1^1(a)) = m_1; \qquad a \in G.$$

By condition 7, at some point $a' \in G$,

$$|\varphi_1(a')| = \left|\sum_{k=1}^{m_1} p_1^k(a')\, \tilde{f}_1^k(q_1^k(a'))\right| \ge c\mu.$$

Let $[a', a''] \subset G$ be a segment of a level line of the function q_1^k such that

$$\omega(\alpha) = \omega(h_1[a', a'']) \le c[2m_1]^{-1}.$$

On $[a', a'']$, $\varphi_1(x_1, x_2)$ preserves a constant sign, and for $(x_1, x_2) \in [a', a'']$,

$$|\varphi_1(x_1, x_2)| \ge \tfrac{1}{2}c\mu.$$

Consequently,

$$\left|\int_{s\in[a',a'']} \varphi_1(s)\, ds\right| \ge \tfrac{1}{2}c\mu\alpha.$$

Let $[b', b'']$ be a segment of one of the lines $\{e_{i,j}\}$ such that

$$d_1(a' \cup b') \le 3\beta \quad \text{and} \quad d_1(a'' \cup b'') \le 3\beta.$$

By the definition of F_1 it follows that

$$\left|\int_{s\in[b',b'']} \varphi_1(s)\, ds\right| \le c_8\varepsilon\alpha',$$

where

$$\alpha' = h_1([b', b'']).$$

Hence

$$\left|\int_{s\in[a',a'']} \varphi_1(s)\,ds - \int_{s\in[b',b'']} \varphi_1(s)\,ds\right| \geq \tfrac{1}{2}c\mu\alpha - c_8\varepsilon\alpha'.$$

Assuming that β is sufficiently small in comparison with α, we obtain $\alpha \sim \alpha'$, and from Lemma 2

$$\left|\int_{s\in[a',a'']} \varphi_1(s)\,ds - \int_{s\in[b',b'']} \varphi_1(s)\,ds\right| \leq c_1(\alpha\varepsilon + \mu\alpha\omega(3\beta) + 3\mu\beta).$$

Thus,

$$c_1(\varepsilon\alpha + \alpha\mu\omega(3\beta) + 3\mu\beta) \geq \tfrac{1}{2}c\mu\alpha - c_8\varepsilon\alpha'.$$

From the last inequality we obtain $\mu \leq c_7\varepsilon$, and then, from Theorem 2 ([1]), we obtain

$$\mathcal{H}^{\delta}_{\theta\varepsilon}(F_\nu) \leq \frac{c_9}{\delta}\left(\frac{M}{\theta\varepsilon}\right)^2.$$

Consequently,

$$\mathcal{H}^{\delta}_{\theta\varepsilon}(F) \leq \left(\frac{\mu}{\theta\varepsilon}\right)^2 \frac{c_9}{\delta} + \log r \leq c_4\left(\frac{1}{\theta^2\delta} + \log\frac{m}{\varepsilon}\right).$$

The lemma is proved.

Denote by $\Phi(\rho,\mu,\varepsilon)$ the set of functions $f(z_1,z_2)$ analytic in the domain $G(\rho,z_1,z_2)$, bounded in G by the constant $\mu > 0$, real in the plane $y_1 = y_2 = 0$, and bounded on this plane by the constant $\varepsilon > 0$.

Lemma 4. *For any positive numbers ρ, μ, $\delta \leq \delta_0(G)$ (G as in Lemma 1),*

$$\varepsilon = e^{-(\rho/\delta)^2}\mu$$

and for some $\theta = \theta(\rho,\mu)$ (θ does not depend on δ), the inequality

$$\mathcal{H}^{\delta}_{\theta\varepsilon}(\Phi(\rho,\mu,\varepsilon)) \geq \delta^{-2}c(G)\,\mathrm{mes}_2(G)$$

holds, where $c(G)$ is a constant determined by the domain G.

Using Lemmas 3 and 4, and assuming that ρ is sufficiently small, it is no longer difficult to prove the theorem with the aid of Lemma 1.

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CITED LITERATURE

1. A. G. Vitushkin, DAN, **156**, No. 5 (1964).

Note: Figure translations are in progress. See original paper for figures.