GENERAL CALCULATION OF MATRIX ELEMENTS FOR POLARIZED VECTOR PARTICLES

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1964

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Abstract

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GENERAL CALCULATION OF MATRIX ELEMENTS FOR POLARIZED VECTOR PARTICLES

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(Presented by Academician V. A. Fock, 12 XII 1963)

In paper (^{1}), on the basis of the general method of projection operators in the theory of elementary particles (^{2}), compact matrix expressions were found for 10-dimensional wave functions describing the states of a vector meson ($m \neq 0$) with a given value of the 4-momentum $p = (\mathbf{p}, ip_0)$ and spin projections equal to ± 1. With their aid, in (^{1}) a covariant method was developed for the direct calculation of matrix elements of longitudinally polarized particles with spin 1.

Below a generalization of the indicated method is given for the case of arbitrary spin states of a vector particle, and on this basis a general calculation of matrix elements of vector particles is carried out. As an illustration, the differential cross sections are calculated for the scattering of a polarized vector meson by a scalar particle and by a Coulomb center.

In accordance with (^{2}), the projection dyad matrices $\Lambda^{(r)}(p)$, which determine the states of a vector particle with 4-momentum p and spin projection $r = \pm 1, 0$, have the form (see (^{1})) ($\hbar = c = 1$)

$$\Lambda^{(\pm)}(p) = \mp \frac{1}{4m^2} \sigma_s(\sigma_s \pm 1)\hat{p}(\hat{p} \mp im) =$$

$$= \mp \frac{1}{2m^2} \left(\hat{e}^{(\mp)}\right)^2 \left(\hat{e}^{(\pm)}\right)^2 \hat{p}(\hat{p} \mp im) = \psi^{(\pm 1)}(\pm p) \cdot \bar{\psi}^{(\pm 1)}(\pm p); \tag{1}$$

$$\Lambda^{(0)}(p) = \pm \frac{1}{2m^2}(\sigma_s^2 - 1)\hat{p}(\hat{p} \mp im) = \psi^{(0)}(\pm p) \cdot \bar{\psi}^{(0)}(\pm p), \tag{2}$$

where $(\psi_1 \cdot \bar{\psi}_2)_{mn} = \psi_{1m}\bar{\psi}_{2n}$, $\bar{\psi} = \psi^* \eta$, $\eta = 2\beta_4^2 - 1$. Here $\bar{p} = p_\mu \beta_\mu$, β_μ are 10×10 Duffin-Kemmer matrices, and

$$\sigma_s = \frac{1}{m} \delta_{\mu\nu\rho\sigma} p_\mu s_\nu \beta_\rho \beta_\sigma = \hat{e}^{(-)}\hat{e}^{(+)} - \hat{e}^{(+)}\hat{e}^{(-)} \tag{3}$$

is the spin-projection operator in the covariant notation of (^{1}). The spin vector

$$s = (\mathbf{s}, is_0) = \left(\frac{p_0}{m}\frac{\mathbf{p}}{|\mathbf{p}|}, i\frac{|\mathbf{p}|}{m}\right)$$

and the circular vectors $e^{(\pm)}$ are connected by the relations (see the case of the electromagnetic field (3))

$$\left(e^{(\pm)}\right)^2 = 0, \qquad e^{(+)}e^{(-)} = 1, \qquad e^{(\pm)}s = 0, \qquad s^2 = 1.$$

Following (1), after simple transformations, from (1) and (2) one can obtain a general expression for the wave function of a vector meson in the form

$$\psi^{(r)}(p) = \frac{1}{m\sqrt{2}}(\hat{p} - im)a_\mu^{(r)}\varepsilon^{\mu 1}, \tag{4}$$

where ε^{AB} are 10×10 matrices with a single matrix element different from zero and equal to 1 at the intersection of the A-th row and the B-th column;

$$\begin{aligned} a^{(r)} &= e^{(-)} \quad \text{for } r = +1, \\ a^{(r)} &= e^{(+)} \quad \text{for } r = -1, \\ a^{(r)} &= s \qquad \text{for } r = 0. \end{aligned} \tag{5}$$

Table 1

Matrix elements for polarized vector particles ($a_1 = a_1^{(r_1)},\ a_2 = a_2^{(r_2)}$)

Q	$M^{r_1 \to r_2}_{p_1 \to p_2} = \frac{1}{2m^2}\,\mathrm{Sp}\{Q(\hat{p}_1 - im)(\hat{a}_2\hat{a}_1 - a_1a_2)P(\hat{p}_2 - im)\}$
$I = \bar{P} + P$	$\frac{1}{2m^2}\{(p_1a_2)(p_2a_1) - (a_1a_2)[(p_1p_2) - m^2]\}$
β_μ	$\frac{i}{2m}\{(a_1a_2)(p_{1\mu} + p_{2\mu}) - (p_1a_2)a_{1\mu} - (p_2a_1)a_{2\mu}\}$
$\beta_\mu\beta_\nu$	$\frac{1}{2m^2}\{(p_1a_2)p_{2\mu}a_{1\nu} + (p_2a_1)a_{2\mu}p_{1\nu} - (p_1p_2)a_{2\mu}a_{1\nu} \quad -(a_1a_2)(p_{2\mu}p_{1\nu} - m^2\delta_{\mu\nu}) - m^2a_{1\mu}a_{2\nu}\}$
$\beta_\mu\beta_\nu\beta_\rho$	$\frac{i}{2m}\{(a_1a_2)(p_{2\mu}\delta_{\nu\rho} + p_{1\rho}\delta_{\mu\nu}) - (p_1a_2)a_{1\rho}\delta_{\mu\nu} \quad -(p_2a_1)a_{2\mu}\delta_{\nu\rho} - p_{2\mu}a_{1\nu}a_{2\rho} + a_{2\mu}a_{1\nu}p_{2\rho} - a_{1\mu}a_{2\nu}p_{1\rho} + p_{1\mu}a_{2\nu}a_{1\rho}\}$

Q	$M^{r_1\to r_2}_{p_1\to p_2} = \frac{1}{2m^2}\,\mathrm{Sp}\{Q(\hat{p}_1 - im)(\hat{a}_2\hat{a}_1 - a_1a_2)P(\hat{p}_2 - im)\}$
$\beta_\mu\beta_\nu\beta_\rho\beta_\sigma$	$\frac{1}{2m^2}\{(a_1a_2)(m^2\delta_{\mu\nu}\delta_{\rho\sigma} - p_{2\mu}p_{1\sigma}\delta_{\nu\rho}) \quad +[(p_1a_2)p_{2\mu}a_{1\sigma} + (p_2a_1)a_{2\mu}p_{1\sigma} - (p_1p_2)a_{2\mu}a_{1\sigma}]\delta_{\nu\rho} \quad +(p_{2\mu}a_{2\rho} - a_{2\mu}p_{2\rho})a_{1\nu}p_{1\sigma} + (a_{2\mu}p_{2\rho} - p_{2\mu}a_{2\rho})p_{1\nu}a_{1\sigma} \quad -m^2(a_{1\mu}a_{2\nu}\delta_{\rho\sigma} + a_{2\sigma}a_{1\rho}\delta_{\mu\nu} - a_{2\nu}a_{1\rho}\delta_{\mu\sigma})\}$
$\eta_5\beta_\mu\beta_\nu\beta_\rho$	$\frac{i}{2m}\{(a_1a_2)(p_{2\mu}\delta_{\nu\rho} - p_{1\rho}\delta_{\mu\nu}) + (p_1a_2)a_{1\rho}\delta_{\mu\nu} \quad -(p_2a_1)a_{2\mu}\delta_{\nu\rho} - p_{2\mu}a_{1\nu}a_{2\rho} + a_{2\mu}a_{1\nu}p_{2\rho} \quad +a_{1\mu}a_{2\nu}p_{1\rho} - p_{1\mu}a_{2\nu}a_{1\rho}\}$
$\eta_5\beta_\mu\beta_\nu$	$\frac{1}{2m^2}\{(p_1a_2)p_{2\mu}a_{1\nu} + (p_2a_1)a_{2\mu}p_{1\nu} - (p_1p_2)a_{2\mu}a_{1\nu} \quad -(a_1a_2)(p_{2\mu}p_{1\nu} + m^2\delta_{\mu\nu}) + m^2a_{1\mu}a_{2\nu}\}$
$\eta_5\beta_\mu$	$\frac{i}{2m}\{(a_1a_2)(p_{2\mu} - p_{1\mu}) + (p_1a_2)a_{1\mu} - (p_2a_1)a_{2\mu}\}$
$\eta_5 = \bar{P} - P$	$\frac{1}{2m^2}\{(p_1a_2)(p_2a_1) - (a_1a_2)[(p_1p_2) + m^2]\}$

Represent the matrix element $M^{r_1\to r_2}_{p_1\to p_2}$, which connects the initial state $\psi^{(r_1)}(p_1)$ and the final state $\psi^{(r_2)}(p_2)$ of the vector particle, in the form

$$M^{r_1\to r_2}_{p_1\to p_2} = \bar{\psi}^{(r_2)}(p_2)Q\psi^{(r_1)}(p_1) = \mathrm{Sp}\{Q\psi^{(r_1)}(p_1)\bar{\psi}^{(r_2)}(p_2)\}, \tag{6}$$

where Q is the vertex operator determining the character of the interaction. Then, using the representation of the wave functions in the form (4) and taking into account the relations

$$e^{(\pm)*} = e^{(\mp)}, \qquad e_4^{(\pm)*} = -e_4^{(\mp)}, \qquad S^* = S, \qquad S_4^* = -S_4,$$

$$\varepsilon^{ik}\eta = \varepsilon^{ik}, \qquad \varepsilon^{i4}\eta = -\varepsilon^{i4}, \qquad \varepsilon^{\mu\nu} = (\delta_{\mu\nu} - \beta_\mu\beta_\nu)(3 - \beta^2)$$

after simple transformations we obtain (cf. (1))

$$M^{r_1\to r_2}_{p_1\to p_2} = \frac{1}{2m^2}\,\mathrm{Sp}\{Q(\hat{p} - im)[\hat{a}_2^{(r_2')}\hat{a}_1^{(r_1)} - a_1^{(r_1)}a_2^{(r_2)}](3 - \beta^2)(\hat{p}_2 - im)\}, \tag{7}$$

where $r'_2 = -r_2$ for $r_2 = \pm 1$; $r'_2 = r_2$ for $r_2 = 0$.

Table 1 collects expressions for the matrix elements for Q taken in the form

$$Q = 1,\ \beta_\mu,\ \beta_\mu\beta_\nu,\ \beta_\mu\beta_\nu\beta_\rho,\ \beta_\mu\beta_\nu\beta_\rho\beta_\sigma, \tag{8}$$

$$\eta_5\beta_\mu\beta_\nu\beta_\rho,\ \eta_5\beta_\mu\beta_\nu,\ \eta_5\beta_\mu,\ \eta_5,$$

where the matrix $\eta_5 = P - \mathbf{P}$ is the representation of the operator of total spatial reflection in the 10-dimensional space, while the operators $P = 3 - \beta^2$ and $\mathbf{P} = \beta^2 - 2$ $(\beta_\mu \mathbf{P} - P\beta_\mu = 0)$ are projection operators that single out, respectively, the vector (4-dimensional) and tensor (6-dimensional) parts of the 10-dimensional Duffin–Kemmer space (^{4}).

The calculation of matrix elements is carried out with the aid of the general formulas obtained in (^{4}) for traces of products of the matrices β_μ on P and $\mathbf{P}$. Let us note that the set of vertex operators Q (8), for which the matrix elements have been calculated, covers all linearly independent elements of the basis of the Duffin–Kemmer matrix algebra (cf. (^{5})). Therefore Table 1 makes it possible at once to write down the expression for the matrix element of any process of interaction of vector particles in the form of a linear combination of the tabulated data and thus to unify and simplify to a considerable extent the calculations of scattering cross sections of polarized vector particles.

As an example, let us consider the scattering of a polarized vector particle by a scalar particle. Let $\varphi(p'_1)$ and $\varphi(p'_2)$, $\psi^{(r_1)}(p_1)$ and $\psi^{(r_2)}(p_2)$ be the wave functions of the initial and final states of the scalar and vector particles, respectively. Then the matrix element M entering the general formula for the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{4e^4 m^2 {m'}^2}{p_{10}p_{20}p'_{20}p'_{20}}\,\frac{f_2^2}{y}\left|\frac{\partial(p_{20}+p'_{20})}{\partial|\mathbf{p}_2|}\right|^{-1}|M|^2 \tag{9}$$

can be written in the form

$$M = \mathrm{Sp}\{\beta^5_\mu \varphi(p'_1)\bar{\varphi}(p'_2)\}\ \mathrm{Sp}\{\beta^{(10)}_\mu \psi^{r_1}(p_1)\bar{\psi}^{(r_2)}(p_2)\},$$

where $\beta^{(5)}_\mu$ and $\beta^{(10)}_\mu$ are the 5×5 and 10×10 Duffin–Kemmer matrices. Using the data of Table 1 and taking into account that, according to (^{6}),

$$\mathrm{Sp}\{\beta^{(5)}_\mu \varphi(p'_1)\bar{\varphi}(p'_2)\} = \frac{i}{2m'}(p'_{1\mu} + p'_{2\mu}),$$

we immediately obtain

$$M = \frac{1}{4mm'}\{(a_1a_2)(p_1'+p_2')(p_1+p_2)-(a_1p_1'+a_1p_2')(a_2p_1)-(a_2p_1'+a_2p_2')(a_1p_2)\}. \tag{10}$$

For the purpose of further simplification, let us pass to the center-of-mass system. In addition, let us suppose that the circular vectors $e^{(\pm)}$ are taken to be three-dimensional, i.e. $e_4^{(\pm)} = 0$. Then, using the permissible arbitrariness in the choice of mutually orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, orthogonal to the direction of motion of the particle and related to $e^{(\pm)}$ by the relations

$$e^{(\pm)} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2),$$

we shall assume that the vectors $\mathbf{e}_1$ before and after scattering lie in the scattering plane of the particles, while the vectors $\mathbf{e}_2$ are orthogonal to this plane. In this case, considering all possible cases of initial and final polarizations of the vector particle, i.e. taking different combinations of the vectors $a_1^{(r_1)}$ and $a_2^{(r_2)}$ (see (5)) in the general formula (9), we obtain the following expressions for the matrix elements determining the differential cross sections:

$$M^{++} = M^{--} = \frac{1}{2mm'}[p_0p_0' + \mathbf{p}^2](\cos\theta + 1), \tag{11}$$

$$M^{+-} = M^{-+} = \frac{1}{2mm'}p_0p_0'(\cos\theta - 1), \tag{12}$$

$$M^{\pm0} = M^{0\pm} = \frac{1}{2mm'}\left\{\frac{1}{m\sqrt{2}}\left[(p_0^2 + m^2)p_0' + p_0\mathbf{p}^2\right]\right\}\sin\theta, \tag{13}$$

$$M^{00} = \frac{1}{2mm'}[2p_0p_0'\cos\theta + \mathbf{p}^2(\cos\theta + 1)]. \tag{14}$$

Hence, for unpolarized particles we shall have

$$|M|^2 = \frac{1}{3}\sum|M^{r_1r_2}|^2 = \left(\frac{p_0p_0'}{mm'}\right)^2\left(1 + \frac{1}{6}\frac{\mathbf{p}^4}{m^2p_0^2}\sin^2\theta\right) +$$

$$+\frac{1}{m'^2}\left(\frac{\mathbf{p}}{m}\right)^2\left[(2p_0p_0' + \mathbf{p}^2)\left(\cos^2\frac{\theta}{2} + \frac{1}{6}\frac{\mathbf{p}^2}{m^2}\sin^2\theta\right) - \frac{1}{12}\mathbf{p}^2\sin^2\theta\right]. \tag{15}$$

Using the formulas obtained, and putting $m' \to \infty$, we immediately find the differential cross sections for scattering of a vector particle by a Coulomb center

$$\frac{d\sigma^{\pm\pm}}{d\Omega} = \sigma_R(\cos^2\theta + 1) \tag{16}$$

$$\frac{d\sigma^{\pm 0}}{d\Omega} = \frac{d\sigma^{0\pm}}{d\Omega} = \sigma_R\left(\frac{p_0^2 + m^2}{2mp_0}\right)^2 \sin^2\theta, \tag{17}$$

$$\frac{d\sigma^{00}}{d\Omega} = \sigma_R \cos^2\theta, \tag{18}$$

$$\frac{d\sigma}{d\Omega} = \sigma_R\left(1 + \frac{1}{6}\frac{\mathbf{p}^4}{m^2 p_0^2}\sin^2\theta\right), \tag{19}$$

where

$$\sigma_R = \frac{1}{4}e^4\frac{p_0^2}{\mathbf{p}^4\sin^4(\theta/2)},$$

θ is the scattering angle.

Formulas (16) and (18) coincide with those given in (7), whereas for case (17) in paper (7) the factor $\sin^2\theta$ was omitted.

In conclusion, we note that the proposed method of calculation may prove useful in calculations of processes involving the deuteron, regarded as a particle with spin 1.

The authors express their deep gratitude to Prof. F. I. Fedorov for guidance and useful advice.

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Received
10 XII 1963

REFERENCES

[1] A. A. Bogush, DAN, **149**, 1286 (1963).
[2] F. I. Fedorov, ZhETF, **35**, 493 (1958).
[3] A. A. Bogush, L. G. Moroz, Vestsi AN BSSR, Ser. fiz.-tekhn. navuk, No. 4, 60 (1961).
[4] A. A. Bogush, F. I. Fedorov, Dokl. AN BSSR, **6**, 81 (1962).
[5] A. A. Insarov, Kh. Byuglian, Tr. Inst. fiz. i astr. AN ESSR, No. 19, 132 (1962).
[6] A. A. Bogush, Dokl. AN BSSR, **7**, 520 (1963).
[7] S. A. Bludman, J. A. Young, Phys. Rev., **126**, 303 (1962).