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Abstract

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ON SOME PERTURBATIONS OF A CLOSED LINEAR OPERATOR

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Consider a vector space $\mathfrak{X}$ and a linear operator T acting in it. Let $\mathfrak{D}_T$, $\mathfrak{R}_T$, and $\mathfrak{Z}_T$ denote, respectively, the domain, the range, and the set of all zeros of the operator T. Introduce the linear sets

$$\mathfrak{M}_T = \bigcup_{n=1}^{\infty} \mathfrak{R}_{T^n} \quad \text{and} \quad \mathfrak{N}_T = \bigcup_{n=1}^{\infty} \mathfrak{Z}_{T^n}.$$

Obviously, $\mathfrak{M}_T \subset \mathfrak{D}_T$ and

$$T(\mathfrak{D}_T \cap \mathfrak{M}_T) \subset \mathfrak{M}_T, \qquad T(\mathfrak{N}_T) \subset \mathfrak{N}_T.$$

Put

$$\mathfrak{Z}_0 = \mathfrak{Z}_T, \qquad \mathfrak{Z}_n = \mathfrak{Z}_0 \cap \mathfrak{R}_{T^n}$$

$$(n = 1, 2, \ldots), \qquad \mathfrak{Z}_\omega = \mathfrak{Z}_0 \cap \mathfrak{M}_T \left(= \bigcap_{n=1}^{\infty} \mathfrak{Z}_n \right).$$

Theorem 1. *If, beginning with some number m, all $\mathfrak{Z}_n$ coincide, then*

$$T(\mathfrak{D}_T \cap \mathfrak{M}_T) = \mathfrak{M}_T.$$

Proof. Take an arbitrary element y of $\mathfrak{M}_T$. Then $y \in \mathfrak{R}_{T^{n+1}}$ for every n. Denote by y_n some preimage of the element y contained in $\mathfrak{R}_{T^n}$ $(n = 1, 2, \ldots)$. This means that $y_n \in \mathfrak{D}_T \cap \mathfrak{R}_{T^n}$ and $Ty_n = y$. The difference

$$z_{mk} = y_m - y_{m+k}$$

is contained in $\mathfrak{R}_{T^m}$, and $Tz_{mk} = 0$ $(k = 1, 2, \ldots)$. Consequently, $z_{mk} \in \mathfrak{Z}_m$. But $\mathfrak{Z}_m = \mathfrak{Z}_{m+1} = \cdots = \mathfrak{Z}_\omega$, whence it is clear that $z_{mk} \in \mathfrak{Z}_{m+k}$. Thus

$$y_m = z_{mk} + y_{m+k} \in \mathfrak{R}_{m+k}.$$

Since this is true for every k and $y_m \in \mathfrak{D}_T$, we have $y_m \in \mathfrak{D}_T \cap \mathfrak{M}_T$. Hence (since $Ty_m = y$) we conclude that

$$T(\mathfrak{D}_T \cap \mathfrak{M}_T) = \mathfrak{M}_T.$$

Let α_T be the dimension of $\mathfrak{Z}_T$, and β_T the dimension of the space complementary to $\mathfrak{R}_T$.

Theorem 2. *If at least one of the numbers α_T or β_T is finite, then, beginning with some number, all $\mathfrak{Z}_n$ are identical.*

Proof. If α_T is finite, then the assertion is obvious (for the $\mathfrak{Z}_n$ do not increase and are contained in $\mathfrak{Z}_T$).

Consider the case where β_T is finite. We shall prove the theorem by contradiction, i.e., suppose that there exists an increasing sequence

$$k_1,\ k_2, \dots$$

such that

$$\mathfrak{Z}_0 = \mathfrak{Z}_1 = \dots = \mathfrak{Z}_{k_1} \neq \mathfrak{Z}_{k_1+1} = \mathfrak{Z}_{k_1+2} = \dots = \mathfrak{Z}_{k_2} \neq \mathfrak{Z}_{k_2+1} = \cdots.$$

Take

$$z_i \in \mathfrak{Z}_{k_i}, \qquad z_i \notin \mathfrak{Z}_{k_i+1} \quad (i = 1, 2, \dots).$$

Then $z_i \in \mathfrak{R}_{k_i}$, $z_i \notin \mathfrak{R}_{k_i+1}$. To each z_i assign an x_i such that

$$T^{k_i}x_i = z_i.$$

The sequence $x_1, x_2, \dots$ is linearly independent. Indeed, let

$$x = \sum_{i=1}^{r} \alpha_i x_i = 0.$$

Then

$$T^{k_r}x = \alpha_{rT}^{k_r}x_r = \alpha_r z_r = 0,$$

and since $z_r \neq 0$, it follows that $\alpha_r = 0$. Similarly we obtain

$$\alpha_{r-1} = 0, \dots, \alpha_1 = 0.$$

Consequently, $x_1, \dots, x_r$ are linearly independent for every r. Denote by $\mathfrak{L}_r$ the linear span of the elements $x_1, \dots, x_r$, and show that

$$\mathfrak{L}_r \cap \mathfrak{R}_1 = 0.$$

Let $x \in L_r \cap \mathfrak{R}_1$, i.e.,

$$\sum_{i=1}^{r} \alpha_i x_i \in \mathfrak{R}_1.$$

Then

$$x = Tx', \qquad \alpha_r z_r = T^{k_r}x = T^{k_r+1}x' \in \mathfrak{R}_{k_r+1}.$$

But $z_r \notin \mathfrak{R}_{k_r+1}$, and therefore $\alpha_r = 0$. Similarly,

$$\alpha_{r-1} = 0, \dots, \alpha_1 = 0.$$

Thus $x = 0$, i.e.,

$$\mathfrak{L}_r \cap \mathfrak{N}_1 = 0.$$

This proves (in view of the arbitrariness of r) that outside $\mathfrak{N}_1$ there are infinitely many linearly independent elements, contrary to the assumption that β_T is finite.

Let us note that the finiteness of one of the numbers α_T or β_T is not a condition necessary for all $\mathfrak{Z}_n$, beginning with some index, to coincide. Indeed, let the space $\mathfrak{X}$ be decomposed into the direct sum of two infinite-dimensional subspaces. Then each of the projection operators thereby obtained gives an example of an operator for which both numbers α and β are infinite, while $\mathfrak{Z}_1 = \mathfrak{Z}_2 = \dots = \{0\}$.

Further we shall assume that the space $\mathfrak{X}$ is a Banach space. Let A be a closed linear operator acting in $\mathfrak{X}$. Fixing A, consider some set $\{B\}$ of bounded linear operators ($B : \mathfrak{X} \to \mathfrak{X}$), commuting with A and with one another (commutativity of B with A means that B maps $\mathfrak{D}_A$ into itself and $ABx = BAx$ for $x \in \mathfrak{D}_A$). Denote by $\mathfrak{B}$ the closure of the linear hull of the set $\{B\}$. It is easy to see that all operators belonging to the Banach algebra $\mathfrak{B}$ commute with A and with one another. As $\{B\}$ one may take, for example, the set consisting of the identity operator I alone; then $\mathfrak{B}$ is a one-dimensional space with elements λI. If the operator A is defined on all of $\mathfrak{X}$, then as $\{B\}$ one may take the set $\{I, A, A^2, \dots\}$; then the elements of $\mathfrak{B}$ will be all possible polynomials in A and their limits.

Suppose now that, together with A, $\mathfrak{B}$ is also fixed. In $\mathfrak{B}$ distinguish the set Φ, consisting of operators S such that: 1) the range $\mathfrak{M}_T$ of the operator $T = A_S = A - S$ is closed in $\mathfrak{X}$; 2) at least one of the numbers α_T or β_T is finite (here β_T denotes the dimension of the space complementary to $\mathfrak{M}_T$ in the algebraic sense). Then, by Theorems 2 and 1, for every S in Φ the operator $T = A - S$ maps $\mathfrak{D}_A \cap \mathfrak{M}_T$ onto $\mathfrak{M}_T$. The set Φ is open in $\mathfrak{B}$ (^{1}).

Our subsequent task is to study the sets $\mathfrak{M}_{A_S}$ and $\mathfrak{N}_{A_S}$ as S varies in Φ.

Let A'_S be the restriction of the operator A_S to $\mathfrak{D}_{A'_S} = \mathfrak{D}_A \cap \mathfrak{M}_{A_S}$. It generates, in the quotient space $\mathfrak{M}_{A_S}/\mathfrak{Z}_{A'_S}$, the invertible operator $\widetilde{A}'_S$, defined by the equality $\widetilde{A}'_S \tilde{x} = A_{Sx}$, where $\tilde{x}$ is the coset (element of $\mathfrak{M}_{A_S}/\mathfrak{Z}_{A'_S}$) containing x ($x \in \mathfrak{D}_{A'_S}$). The operator $(\widetilde{A}'_S)^{-1}$ is bounded. Let

$$N'_S = \|(\widetilde{A}'_S)^{-1}\|.$$

Theorem 3. *For every $S_0 \in \Phi$ there exists a neighborhood in Φ such that the inclusions*

$$\mathfrak{M}_{A_{S_0}} \subset \mathfrak{M}_{A_S} \quad \text{and} \quad \mathfrak{N}_{A_S} \cap \mathfrak{M}_{A_{S_0}} \subset \mathfrak{N}_{A_{S_0}}$$

hold.

Proof. Fix an element $y_0 \in \mathfrak{M}_{A_{S_0}}$. The equation $A_{S_0} y = y_0$ has a solution $y_1 \in \mathfrak{M}_{A_{S_0}}$ such that $\|y_1\| < (N'_{S_0} + 1)\|y_0\|$. Similarly, the equation $A_{S_0} y = y_1$ has a solution y_2, for which $\|y_2\| < (N'_{S_0} + 1)\|y_1\|$, etc. Form the series

$$y_S^0 = \sum_{k=0}^{\infty} (S - S_0)^k y_{k+1}$$

and the series

$$y_S^n = \frac{1}{n!} \frac{d^n}{dS^n} y_S^0 \quad (n = 1, 2, \ldots),$$

which all converge in the ball

$$\|S - S_0\| < (N'_{S_0} + 1)^{-1}.$$

Using the closedness of the operator A_{S_0}, its commutativity with every operator S from $\mathfrak{B}$, and the equality $A_{S_0}(\mathfrak{D}_A \cap \mathfrak{M}_{A_{S_0}}) = \mathfrak{M}_{A_{S_0}}$, one can prove that all terms and sums of the series under consideration belong to $\mathfrak{D}_A \cap \mathfrak{M}_{A_{S_0}}$. A direct verification shows that

$$A_S^n y_S^{n-1} = y_0, \quad n = 1, 2, \ldots$$

(here one has to apply the operator A_S to the series term by term, which is permissible because A_S is closed). Consequently, $y_0 \in \mathfrak{M}_{A_S}$. In view of the arbitrariness of the element y_0 from $\mathfrak{M}_{A_{S_0}}$, we obtain the inclusion $\mathfrak{M}_{A_{S_0}} \subset \mathfrak{M}_{A_S}$, where $\|S - S_0\| < (N'_{S_0} + 1)^{-1}$.

The second inclusion is obtained in a similar way (^{1}), by considering the series

$$x_S^0 = \sum_{k=0}^{\infty} (S - S_0)^k x_k \quad \text{and} \quad x_S^n = \frac{1}{n!} \frac{d^n}{dS^n} x_S^0 \quad (n = 1, 2, \ldots),$$

where $x_0, x_1, \ldots$ is a sequence of elements chosen in a definite way from $\mathfrak{M}_{A_{S_0}} \cap \mathfrak{M}_{A_{S_0}}$. The magnitude of the corresponding neighborhood is estimated by the inequality

$$\|S - S_0\| < \frac{1}{2} (N'_{S_0} + 1)^{-1}.$$

Denote by Γ the set of those S in Φ for which the operator A_S has zeros not contained in $\mathfrak{M}_{A_S}$.

Theorem 4. *The set* Γ *is closed in* Φ.

Proof. Suppose the contrary. Then there exists an accumulation point S_0 of the set Γ, belonging to $\Phi \setminus \Gamma$. Let $S_n \in \Gamma$ $(n = 1, 2, \ldots)$ and $\|S_n - S_0\| \to 0$. For each zero x_0 of the operator A_{S_0} construct, as above, the element

$$x_S^0 = \sum_{k=0}^{\infty} (S - S_0)^k x_k,$$

which is a zero of the operator A_S in some neighborhood of the point S_0 and belongs to $\mathfrak{M}_{A_{S_0}} \subset \mathfrak{M}_{A_S}$. Such a construction is possible, since every zero of the operator A_{S_0} belongs to $\mathfrak{M}_{A_{S_0}}$. Without loss of generality, one may assume that all S_n are contained in this neighborhood.

Denote by Ω_n the set of all zeros of the operator A_{S_n} of the form $x^0_{S_n}$, corresponding to all possible $x_0 \in \mathfrak{Z}_{A_{S_0}}$. The sets Ω_n are linear and $\Omega_n \subset \mathfrak{M}_{A_{S_n}}$. Since $\overline{\Omega}_n \subset \mathfrak{Z}_{A_{S_n}}$, $\overline{\Omega}_n \subset \mathfrak{M}_{A_{S_n}}$, and $S_n \in \Gamma$, there is an element z_n in $\mathfrak{Z}_{A_{S_n}}$ such that $\|z_n\| = 1$, $\rho(z_n, \overline{\Omega}_n) > 1/2$. Hence we conclude that $\rho(z_n, \mathfrak{Z}_{A_{S_0}}) > 1/3$, starting from some index. This follows from the fact that $\rho(z_n, \mathfrak{Z}_{A_{S_0}}) = \rho(z_n, K)$, where K is the ball in $\mathfrak{Z}_{A_{S_0}}$ with center at the zero point and radius 2, and from the fact that every point $x_0 \in K$ is arbitrarily little distant from the corresponding point $x^0_{S_n} \in \mathfrak{Z}_{A_{S_n}}$, beginning with some index, uniformly in K. But

$$A_{S_0} z_n = A_{S_0} z_n - A_{S_n} z_n = (S_n - S_0) z_n \to 0$$

as $n \to \infty$, i.e. $y_n = A_{S_0} z_n \to 0$. Consequently,

$$y_n = \widetilde{A}_{S_0} \tilde{z}_n \to 0,$$

where $\widetilde{A}_{S_0}$ is a continuously invertible operator acting in $\mathfrak{X}/\mathfrak{Z}_{A_{S_0}}$. Hence

$$\tilde{z}_n = (\widetilde{A}_{S_0})^{-1} y_n \to 0,$$

in contradiction to the fact that

$$\|\tilde{z}_n\| = \rho(z_n, \mathfrak{Z}_{A_{S_0}}) > 1/3.$$

Theorem 5. *If S runs through the set $G \setminus \Gamma$, where G is some connected component of the set Φ, then the spaces $\overline{\mathfrak{R}}_{A_S}$ and $\mathfrak{M}_{A_S}$ are constant.*

Proof. Let S_1 and S be any two points of $G \setminus \Gamma$, and let F be a polygonal line in $G \setminus \Gamma$ joining S_1 with S. Take on F a finite number of points S_i $(i = 1, 2, \dots, m)$, $S_m = S$, for which

$$\|S_i - S_{i+1}\| < \frac{1}{2}\left(1 + \sup_{S \in F} N'_S\right)^{-1}.$$

The finiteness of $\sup_{S \in F} N'_S$ can be proved by the same method as the analogous assertion in Lemma 2 (^{2}).

On the basis of Theorem 3 (from the estimates for $\|S - S_0\|$) there follow the equalities

$$\mathfrak{M}_{A_{S_i}} = \mathfrak{M}_{A_{S_{i+1}}} \quad \text{and} \quad \overline{\mathfrak{R}}_{A_{S_i}} = \overline{\mathfrak{R}}_{A_{S_{i+1}}}, \quad i = 1, 2, \dots, m-1,$$

whence

$$\mathfrak{M}_{A_{S_1}} = \mathfrak{M}_{A_S}$$

and

$$\overline{\mathfrak{R}}_{A_{S_1}} = \overline{\mathfrak{R}}_{A_S}.$$

Let us note that assertions analogous to Theorems 3 and 5 were obtained in paper (2) for the case when $\mathfrak{B}$ is generated by a single operator I. As for Theorem 4, instead of it we had there a more special result—the isolation of Γ in Φ.

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CITED LITERATURE

[1] M. A. Gol' dman, DAN, 100, No. 2 (1955). [2] M. A. Gol' dman, S. N. Krachkovskii, DAN, 154, No. 3 (1964).

Note: Figure translations are in progress. See original paper for figures.

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