A. YANUSHAUSKAS

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Abstract

Full Text

A. YANUSHAUSKAS

THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION AND A MULTIPLICATION OPERATION FOR HARMONIC FUNCTIONS

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Let C^n be the space of n independent complex variables, R^n the n-dimensional Euclidean space; B a bounded domain of holomorphy in the space C^2 (a domain of holomorphy is understood to mean a domain in which there exists at least one holomorphic function that cannot be continued holomorphically from this domain).

In this note we consider the Cauchy problem for the Laplace equation

$$\Delta f \equiv \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0. \tag{1a}$$

The investigation of the Cauchy problem is carried out in a complex domain. The results obtained for the Cauchy problem are used to introduce a multiplication operation $\circ$ for harmonic functions.

Make the following change of variables in (1a):

$$\xi = x + iy, \qquad \eta = x - iy, \qquad \zeta = z. \tag{2}$$

In these variables equation (1a) is rewritten in the form

$$4\partial^2 f/\partial\xi\,\partial\eta + \partial^2 f/\partial\zeta^2 = 0. \tag{1}$$

Consider the following problem.

Problem K. Find a solution $f(\xi, \eta, \zeta)$ of equation (1) satisfying the conditions

$$f\big|_{\zeta=0} = u(\xi, \eta), \qquad \partial f/\partial\zeta\big|_{\zeta=0} = v(\xi, \eta), \tag{3}$$

where u and v are functions holomorphic in the domain of holomorphy B.

Represent the function $f(\xi, \eta, \zeta)$ as the sum

$$f(\xi, \eta, \zeta) = g(\xi, \eta, \zeta) + h(\xi, \eta, \zeta),$$

where g and h are solutions of equations (1), satisfying the conditions:

$$g\big|_{\zeta=0} = u, \qquad \partial g/\partial\zeta\big|_{\zeta=0} = 0; \qquad h\big|_{\zeta=0} = 0, \qquad \partial h/\partial\zeta\big|_{\zeta=0} = v.$$

As is known ([1]), the functions g and h are represented by the series

$$g = \sum_{n=0}^{\infty}(-1)^n \frac{\zeta^{2n}}{(2n)!}\, 4^n \frac{\partial^{2n} u}{\partial\xi^n \partial\eta^n}, \tag{4a}$$

$$h = \sum_{n=0}^{\infty}(-1)^n \frac{\zeta^{2n+1}}{(2n+1)!}\, 4^n \frac{\partial^{2n} v}{\partial\xi^n \partial\eta^n}. \tag{4b}$$

This representation is valid only in the domain of absolute and uniform convergence of the series. We investigate the domain of convergence.

Lemma 1. If u and v are holomorphic in the bicylinder $D_0 : \{|\xi - \xi_0| < r,\ |\eta - \eta_0| < r\}$, then the series (4a) and (4b) converge absolutely and uniformly in the circle $K_0 : \{\xi = \xi_0,\ \eta = \eta_0,\ |\zeta| < r\}$.

Proof. We prove the assertion of the lemma for the function g. For functions holomorphic in the bicylinder D_0 the estimate ([2])

$$\left|\partial^{2n} u/\partial\xi^n \partial\eta^n\right|_{\xi=\xi_0,\ \eta=\eta_0} \le M (n!)^2 / r^{2n}, \tag{5}$$

is known, where

$$M = \max_{D_0} |u(\xi, \eta)|.$$

Using this estimate, we obtain

$$|g| \le M \sum_{n=0}^{\infty} \frac{4^n (n!)^2}{(2n)!} \left|\frac{\zeta}{r}\right|^{2n}.$$

Since

$$\lim_{n\to\infty} \frac{4^{n+1}[(n+1)!]^2}{(2n+2)!} : \frac{4^n (n!)^2}{(2n)!} = 1,$$

the series

(4a) converges absolutely and uniformly in the disk K_0. The assertion of the lemma for the function h is proved analogously.

Let the point (ξ_0, η_0) belong to the domain of holomorphy B. Consider the maximal bicylinder $D : \{|\xi - \xi_0| < r,\ |\eta - \eta_0| < r\}$ contained in B. By the lemma, the series (4) converge absolutely and uniformly on the set $U(B)$: $\{|\zeta| < r,\ (\xi_0, \eta_0) \in B\}$. The set $U(B)$ contains an open set V in C^3 such that $B \subset V \subset U(B)$ ([3]).

We now take as the domain B the bicylinder $D = D_1 \times D_2$, where D_1 is a domain in the ξ-plane with smooth boundary Γ_1, and D_2 is a domain in the η-plane with smooth boundary Γ_2. The set $\Gamma = \Gamma_1 \times \Gamma_2$ is called the skeleton of the boundary of the bicylinder D. As is known ([2]), any function $u(\xi, \eta)$, holomorphic in D and Hölder-continuous in the closed domain $\overline{D}$, can be represented in the following form:

$$u(\xi, \eta) = -\frac{1}{4\pi^2} \iint_{\Gamma_1\Gamma_2} \frac{u(t, \tau)}{(t - \xi)(\tau - \eta)}\, d\tau\, dt. \tag{6}$$

Substituting (6) into the series (4a) and changing the order of summation and integration, we obtain

$$g(\xi, \eta, \zeta) = -\frac{1}{4\pi^2} \iint_{\Gamma_1\Gamma_2} \frac{1}{(t - \xi)(\tau - \eta)} F\left(1, 1; \frac{1}{2}; -\frac{\zeta^2}{(t - \xi)(\tau - \eta)}\right) u(t, \tau)\, d\tau\, dt,$$

where $F(a, \beta; \gamma; \lambda)$ is Gauss' s hypergeometric function. Analogously we obtain

$$h(\xi, \eta, \zeta) = -\frac{1}{4\pi^2} \iint_{\Gamma_1\Gamma_2} \frac{\zeta}{(t - \xi)(\tau - \eta)} F\left(1, 1; \frac{3}{2}; -\frac{\zeta^2}{(t - \xi)(\tau - \eta)}\right) v(t, \tau)\, d\tau\, dt.$$

Theorem 1. For every bicylinder $D \subset C^2$ with a smooth skeleton of the boundary Γ, there exists a domain of holomorphy $H(D) \subset C^3$ such that the solution of the problem K, holomorphic in $H(D)$, for arbitrary initial data holomorphic in D, is represented in $H(D)$ by the formula

$$f(\xi, \eta, \zeta) = -\frac{1}{4\pi^2} \iint_{\Gamma_1\Gamma_2} \frac{1}{(t - \xi)(\tau - \eta)} \left[F\left(1, 1; \frac{1}{2}; -\frac{\zeta^2}{(t - \xi)(\tau - \eta)}\right) u(t, \tau) + \right.$$
$$\left. + \zeta F\left(1, 1; \frac{3}{2}; -\frac{\zeta^2}{(t - \xi)(\tau - \eta)}\right) v(t, \tau)\right] d\tau\, dt. \tag{7}$$

For any point X on the boundary of the domain $H(D)$ there exists a solution of problem K with initial data holomorphic in D, having a singularity at the point X.

Proof. As is known ([4]), the hypergeometric function $F(1, 1; \gamma; \lambda)$ is single-valued and holomorphic in the λ-plane cut along the ray $[1, \infty)$, and at the point $\lambda = 1$ it has a singularity of order $2 - \gamma$. $f(\xi, \eta, \zeta)$ in (7) is holomorphic everywhere where

$$F\left(1, 1; \frac{1}{2}; -\frac{\zeta^2}{(t - \xi)(\tau - \eta)}\right)$$

and

$$F\left(1,1;\frac{3}{2};-\frac{\zeta^2}{(t-\xi)(\tau-\eta)}\right)$$

are holomorphic. Let Ω be the union of all surfaces

$$Q:\{\zeta^2+(t-\xi)(\tau-\eta)=0,\ (t,\tau)\in\Gamma\}.$$

Consider the connected component E of the complement $C\Omega$ containing an open neighborhood $V\subset U(D)$ of the set D in C^3. This component will be the domain $H(D)$. We shall show that $H(D)$ is a domain of holomorphy. $H(D)$ is the intersection of domains of holomorphy and contains an open set $V\subset U(B)$ in C^3; by Theorem 11.7 of ([2]), $H(D)$ is a domain of holomorphy. The second assertion of the theorem is obvious.

Theorem 2. The set of harmonic functions u regular in the ball

$$(\operatorname{Re}x)^2+(\operatorname{Re}y)^2+(\operatorname{Re}z)^2<R^2$$

and the set of all possible pairs (u,v) of functions holomorphic in the domain

$$D:\{|x+iy|<R,\ |x-iy|<R\}$$

are in one-to-one correspondence.

Proof. Formula (7) in this case takes the form

$$f(\xi,\eta,\zeta)=-\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{(e^{i\varphi}-\xi)(e^{i\psi}-\eta)}\left[F\left(1,1;\frac{1}{2};-\frac{\zeta^2}{(e^{i\varphi}-\xi)(e^{i\psi}-\eta)}\right)du(e^{i\varphi},e^{i\psi})\right.$$
$$\left.+\zeta F\left(1,1;\frac{3}{2};-\frac{\zeta^2}{(e^{i\varphi}-\xi)(e^{i\psi}-\eta)}\right)v(e^{i\varphi},e^{i\psi})\right]d\psi\,d\varphi. \tag{7a}$$

The singularities of the functions $F(1,1;\gamma;-\zeta^2/(e^{i\varphi}-\xi)(e^{i\psi}-\eta))$ $(\gamma=1/2,\ 3/2)$ are situated on the surfaces $\zeta^2+(e^{i\varphi}-\xi)(e^{i\psi}-\eta)=0$. Putting $\operatorname{Im}x=\operatorname{Im}y=\operatorname{Im}z=0$, we obtain $(\operatorname{Re}x)^2+(\operatorname{Re}y)^2+(\operatorname{Re}z)^2+R^2\cos(\varphi+\psi)-R(\operatorname{Re}x)(\cos\varphi+\cos\psi)+R(\operatorname{Re}y)(\sin\varphi-\sin\psi)=0$, $R^2\sin(\varphi+\psi)-R(\operatorname{Re}x)(\sin\varphi+\sin\psi)+R(\operatorname{Re}y)(\cos\psi-\cos\varphi)=0$.

From these equalities we obtain $(\operatorname{Re}x)^2+(\operatorname{Re}y)^2+(\operatorname{Re}z)^2=R^2$. It follows that the function $f(\xi,\eta,\zeta)$ is regular in the ball

$$(\operatorname{Re}x)^2+(\operatorname{Re}y)^2+(\operatorname{Re}z)^2<R^2. \tag{8}$$

Let us show conversely that if the harmonic function f is regular in the ball (8), then u and v are holomorphic in D. Consider two particular cases of problem K:

$$g_\alpha^\beta\big|_{\zeta=0} = \xi^\alpha\eta^\beta, \qquad \partial g_\alpha^\beta/\partial\zeta\big|_{\zeta=0} = 0; \qquad h_\alpha^\beta\big|_{\zeta=0} = 0, \qquad \partial h_\alpha^\beta/\partial\zeta\big|_{\zeta=0} = \xi^\alpha\eta^\beta.$$

The solutions of these problems have the form

$$g_\alpha^\beta = \xi^\alpha\eta^\beta F\left(-\alpha, -\beta; \frac{1}{2}; -\zeta^2/\xi\eta\right), \qquad h_\alpha^\beta = \zeta\xi^\alpha\eta^\beta F\left(-\alpha, -\beta; \frac{3}{2}; -\zeta^2/\xi\eta\right).$$

For natural α and β, g_α^β differs only by a constant factor from the spherical function $r^{2\nu}P_{2\nu}^\mu(\cos\theta)e^{i\mu\varphi}$, and h_α^β differs only by a constant factor from $r^{2\nu+1}P_{2\nu+1}^\mu(\cos\theta)e^{i\mu\varphi}$, where $\nu = \max(\alpha,\beta)$, $\mu = |\alpha-\beta|$, and $P_k^\rho(\omega)$ is the associated Legendre polynomial. As is known ([5]), every harmonic function regular in the ball (8) can be expanded into an absolutely and uniformly convergent series in spherical functions

$$f(x,y,z) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} r^{2\nu}e^{i\mu\varphi}\left[a_\nu^\mu P_{2\nu}^\mu(\cos\theta) + rb_\nu^\mu P_{2\nu+1}^\mu(\cos\theta)\right]. \tag{9}$$

From (9), for $z = 0$ we have

$$f\big|_{z=0} = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} A_\nu^\mu (x+iy)^\nu (x-iy)^{\nu+\mu}, \tag{10}$$

$$\left.\frac{\partial f}{\partial z}\right|_{z=0} = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} B_\nu^\mu (x+iy)^\nu (x-iy)^{\nu+\mu}.$$

The series (10) converge absolutely and uniformly in the circle $|x+iy| < R$, $|x-iy| < R$, $\xi = \eta$, and therefore they will converge for $|\xi| < R$, $|\eta| < R$, i.e. in the domain $D : \{|\xi| < R,\ |\eta| < R\}$. The theorem is proved.

Consider the set of harmonic functions g, regular in the ball (8) and satisfying the conditions

$$g\big|_{\zeta=0} = u(\xi,\eta), \qquad \partial g/\partial\zeta\big|_{\zeta=0} = 0. \tag{11}$$

By Theorem 2, these harmonic functions are in one-to-one correspondence with the functions $u(\xi,\eta)$ holomorphic in the domain $D : \{|\xi| < R,\ |\eta| < R\}$. Denote by $F(D)$ the ring of functions holomorphic in the domain D, and by $G(F)$ the set of harmonic functions satisfying the conditions (11). On the set $G(F)$ one can introduce a ring structure as follows: let $g_1\big|_{\zeta=0} = u_1$, $g_2\big|_{\zeta=0} = u_2$; then $g_3 = g_1 + g_2$ satisfies the conditions $g_3\big|_{\zeta=0} = u_1 + u_2$, $\partial g_3/\partial\zeta\big|_{\zeta=0} = 0$, and the multiplication operation $\circ$ is defined as follows: $g_4 = g_1 \circ g_2$, if $g_4\big|_{\zeta=0} = u_1 \cdot u_2$, $\partial g_4/\partial\zeta\big|_{\zeta=0} = 0$. Endowed with these operations, $G(F)$ is a commutative ring without zero divisors. The identity in $G(F)$ is the function $g \equiv 1$.

We denote by $H(F)$ the set of all harmonic functions regular in the ball (8). On $H(F)$ one can define the structure of a $G(F)$-module as follows: let $f \in H(F)$ and let it satisfy the conditions $f|_{\zeta=0} = u$, $\partial f/\partial\zeta|_{\zeta=0} = v$; then $f_1 = g \circ f$ satisfies the conditions $f_1|_{\zeta=0} = u \cdot u_1$, $\partial f/\partial\zeta|_{\zeta=0} = vu_1$, where $u_1 = g|_{\zeta=0}$. In the ring $G(F)$ one can introduce a norm in the following way: if $g|_{\zeta=0} = u_1$, $g \in G(F)$, then set $\|g\| = \max u_1$. In the $G(F)$-module $H(F)$ one can also introduce a norm in the following way: let $f \in H(F)$; then $f = g + h$, where $g \in G(F)$ and $\partial h/\partial\zeta \in G(F)$; set $\|f\| = \sqrt{\|g\|^2 + \|\partial h/\partial\zeta\|^2}$. It is obvious that the norm thus introduced is consistent with all operations defined on $G(F)$ and on $H(F)$, i.e., these operations are continuous in this norm.

As is known ([2]), every function holomorphic in the bicylinder $D : \{|\xi| < R,\ |\eta| < R\}$ can be represented by a power series absolutely and uniformly convergent in D. Denote by $F_0(D)$ the subring of the ring $F(D)$ consisting of all functions holomorphic in D whose power series converge absolutely in the closed bicylinder $\overline{D}$. Denote by $H(F_0)$ the set of functions harmonic in the ball (8) for which the series in spherical functions converge absolutely in the closed ball. Denote by $G(F_0)$ the subset of functions g from $H(F_0)$ satisfying the condition $\partial g/\partial\zeta|_{\zeta=0} = 0$.

Theorem 3. *If $u, v \in F_0(D)$, then the harmonic function f satisfying the conditions $f|_{\zeta=0} = u$, $\partial f/\partial\zeta|_{\zeta=0} = v$, belongs to $H(F_0)$, and conversely.*

Proof. Let $u = \xi^m\eta^n$, $D_\varepsilon : \{|\xi| < R + \varepsilon,\ |\eta| < R + \varepsilon\}$. Applying formula (7) to u and D_ε, and estimating the integral in (7), we obtain

$$\max_{H(D)} |g_m^n| \le K \max_{\overline{D}} |\xi^m\eta^n|,$$

where

$$K = \max_{H(D)} \left| \frac{1}{[(R+\varepsilon)e^{i\varphi} - \xi][(R+\varepsilon)e^{i\psi} - \eta]} \times \right.$$
$$\left. \times F\left(1, 1; \frac{1}{2}; -\frac{\zeta^2}{[(R+\varepsilon)e^{i\varphi} - \xi][(R+\varepsilon)e^{i\psi} - \eta]}\right) \right|.$$

An analogous estimate can be obtained for the harmonic function h_m^n satisfying the conditions $h_m^n|_{\zeta=0} = 0$, $\partial h_m^n/\partial\zeta|_{\zeta=0} = \xi^m\eta^n$. From these estimates the direct assertion of the theorem follows.

Let us prove the converse assertion of the theorem. Suppose we have

$$f(\xi,\eta,\zeta)=\sum_{n,m=0}^{\infty}a_{mn}\xi^m\eta^n F\left(-m,-n;\frac{1}{2};-\frac{\zeta^2}{\xi\eta}\right)+$$

$$+\zeta\sum_{n,m=0}^{\infty}b_{nm}\xi^m\eta^n F\left(-m,-n;\frac{3}{2};-\frac{\zeta^2}{\xi\eta}\right). \tag{12}$$

By hypothesis, the series (12) converge absolutely for $\zeta=0$, $\xi=\eta$, $|\xi|=R$. By Abel' s theorem ([2]) these series converge absolutely for all $|\xi|\leq R$, $|\eta|=R$.

Corollary. *The normed rings $F_0(D)$ and $G(F_0)$ are isomorphic.*

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.